Exploring Asymmetrical Results in Mathematics:
Set Asymmetries in the Real Numbers

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A quote attributed to Isaac Asimov goes "The most exciting phrase to hear in *science* - the one that heralds new discoveries- is not *"Eureka!"* but *"That's funny..."*. With apologies to Archimedes, substitute *mathematics* for *science*.

There are many "that's funny" results in mathematics – this is one of them.

A.L. Cauchy
1789-1857

The 19th century saw an increased interest in mathematical rigor. Mathematicians like Cauchy, Weierstrass, and others worked on defining mathematical concepts like limit, continuity, differentiation, convergence and integration more rigorously. This resulted in a flood of pathological (?) functions and examples aimed at understanding and testing the understanding of the more rigorous definitions. Weierstrass's 1872 example of a continuous nowhere differentiable function is a prime example.

K. Weierstrass
1815-1897

Out of this milieu comes one of my favorite "that's funny" results in mathematics: the existence of a function (*f(x)* below) which is continuous at all irrational points and discontinuous at all rational points (while no function seems to exist which is the other way around: discontinuous at irrational points while continuous at rational points).

$$f(x) = \begin{cases} 1/q & if\ x = p/q\ for\ p\ and\ q\ coprime \\ 0 & if\ x\ is\ irrational \end{cases}$$

J. L. Dirichlet
1805-1859

Refining the definition of function was another fallout from the 19th century as mathematicians continued to extend the boundaries of mathematical understanding. Where previously functions were loosely defined in terms of "analytic expressions", Dirichlet in 1837 gave us what has become the modern definition of function, essentially defining it to be a well-defined correspondence between domain and co-domain,

Though it seems counter-intuitive that the above function would be continuous at irrational points (shouldn't it be discontinuous everywhere?), showing continuity at irrational points is a nicely done using a straight-forward δ –ε proof (which incidentally demonstrates the power of this "new" definition for limit and continuity). Kudos to Cauchy and Weierstrass!

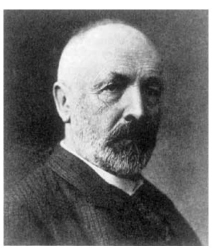
Proof: Restricting our attention to the interval (0, 1) let x_0 be an irrational number on (0, 1) so $f(x_0) = 0$. Then given any $\varepsilon > 0$ choose an integer r such that $1/r < \varepsilon$, Claim there are only finitely many rational numbers p/q on the interval (0,1) whose denominators q are less or equal to r (i.e. $q \le r$)

so $\frac{1}{q} \geq \frac{1}{r}$. Choose δ to be the minimal distance between x_0 and the *finite* set of all rational numbers $\frac{p}{q}$ on (0, 1) such that $q \leq r$. Therefore if $x = \frac{p}{q}$ is any rational number on the interval $(x_0 - \delta, x_0 + \delta)$ then we know $q > r$, hence $|f(x) - 0| = \frac{1}{q} < \frac{1}{r} < \varepsilon$. If x is irrational, $f(x) = 0$ so trivially $|f(x) - 0| = 0 < \varepsilon$ and continuity at irrational points follows.

Showing *discontinuity* at rational points is simply observing that given any rational number $\frac{p}{q}$ one can find an irrational number arbitrarily close to $\frac{p}{q}$. **QED**

Observe that irrational numbers in some sense *avoid* rational numbers $\frac{p}{q}$ with denominators q greater than r for some integer *r*, but the opposite is not true. Irrational numbers cluster around rational numbers.

But why the asymmetry? Perhaps the answer can be understood in showing the rational numbers are countably infinite (they can be put into one-to-one correspondence with the natural numbers) while set of irrational numbers is not countably infinite (they cannot be put into a one-to-one correspondence with the natural numbers) or *non-denumerable*. So in some well-defined mathematical sense, the set of irrational numbers is *larger* than the set of rational numbers; there is an asymmetry here.


G. Cantor
1845-1918

This observation derives from Cantor's 1874 result proving that the set of real numbers $\mathbb{R}$ are non-denumerable in that they cannot be put into a 1:1 correspondence with the natural numbers $\mathbb{N}$.

Specifically

I. The rational numbers $\mathbb{Q}$ are countably infinite in that they can be put into a one-to-one correspondence with the natural numbers $\mathbb{N}$.

II. If A and B are countably infinite sets, their union $A \bigcup B$ is also countably infinite.

III. If the set of irrational numbers $\overline{\mathbb{Q}} = \mathbb{I}rr$ were countably infinite, the union of the rational numbers $\mathbb{Q}$ and the irrational numbers $\overline{\mathbb{Q}} = \mathbb{I}rr$ which make up the set of real numbers $\mathbb{R}$ would be countably infinite (by II) which contradicts the non-denumerability of the real numbers $\mathbb{R}$

Probing further, another "that's funny" asymmetry in mathematics concerns the denumerability of the algebraic numbers (a superset of the rational numbers), numbers which are roots of polynomials with integer coefficients.

Establishing a one-to-one correspondence between the algebraic numbers and natural numbers is done by defining the *height* of a polynomial with integer coefficients as follows

$$H\left(a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0\right) = \sum_{k=0}^{n} |a_k| + n - 1.$$

For example, $H\left(x^2-x-1\right)=4$ while *set of algebraic numbers* with height 4 is

$$\left\{a \mid p_n(a)=0 \text{ and } H\left(p_n(x)\right)=4\right\}=\left\{\pm 3, \pm \tfrac{1}{3}, \pm\sqrt{2}, \pm \tfrac{1}{\sqrt{2}}, \pm\phi, \pm \tfrac{1}{\phi}\right\}$$

if we disallow algebraic numbers obtained from polynomials of smaller degree (note that $\phi=\frac{1+\sqrt{5}}{2}$ is the so-called golden ratio). Since the set of polynomials with integer coefficients for a given height is finite, it's not difficult to establish a one-to-one correspondence between the set of all algebraic numbers and the natural numbers. Non-algebraic numbers called *transcendental* numbers are non-denumerable using the same argument that showed the irrationals are non-denumerable.

J. Liouville
1809-1882

This asymmetry (?) related to the size of the algebraic versus transcendental numbers set is particularly funny since proving that a number is transcendental is difficult to do.

In 1844 Liouville constructed a class of transcendental numbers using continued fractions. In 1851 he proved the number

$$l_0=\sum_{k=1}^{\infty}\frac{1}{10^{k!}}=\frac{1}{10}+\frac{1}{10^2}+\frac{1}{10^6}+\frac{1}{10^{24}}+\ldots=0.110001000000000000000000100..$$

was transcendental using a simple (?) inequality necessary for irrational algebraic numbers.

Liouville's Inequality: If x_0 is an irrational algebraic number with minimal degree polynomial $p_n\left(x\right)=a_nx^n+a_{n-1}x^{n-1}+\ldots+a_1x+a_0$ having integer coefficients, then there is a positive number A such that if p/q is a rational number in the interval $\left[x_0-1, x_0+1\right]$ then $\left|p/q-x_0\right|\geq\frac{1}{Aq^n}$

That is, irrational algebraic numbers in some sense *avoid* rational numbers similar (?) to irrational numbers *avoiding* (?) certain types of rational numbers as mentioned above.

The proof of this inequality is surprisingly simple. After demonstrating that minimal degree polynomials for irrational algebraic numbers have no rational roots (otherwise they are not minimal degree), Liouville used the Mean Value Theorem to obtain the inequality

$$\left|\frac{p_n\left(p/q\right)-p_n\left(x_0\right)}{p/q-x_0}\right|=\left|p_n'\left(c\right)\right|\leq A$$

for some c on the interval $\left[x_0-1, x_0+1\right]$ where $p_n'\left(x\right)$ being a polynomial on a closed interval is bounded by some constant A.

Since $p_n(x_0)=0$ it follows that $\left|q^n \cdot p_n\left({}^{p}\!/_{q}\right)\right| \le A\cdot q^n \left|{}^{p}\!/_{q} - x_0\right|$. However since $p_n(x)$ has no rational roots and the coefficients of $p_n(x)$ are integers, if we multiply $p_n\left({}^{p}\!/_{q}\right)$ by q^n, $\left|q^n \cdot p_n\left({}^{p}\!/_{q}\right)\right|$ is a non-zero integer greater than or equal to 1. Therefore $1 \le A\cdot q^n \left|{}^{p}\!/_{q} - x_0\right|$ so the result follows:

$$\frac{1}{A\cdot q^n} \le \left|{}^{p}\!/_{q} - x_0\right|$$

Liouville then went on to show that his number $l_0 = \sum_{k=1}^{\infty}\frac{1}{10^{k!}}$ did not satisfy this inequality so therefore it was not algebraic but transcendental.

Starting with his inequality and for any $m > n$ let

$${}^{p}\!/_{q} = \sum_{k=1}^{m}\frac{1}{10^{m!}} = \frac{10^{m!-1}+10^{m!-2}+\ldots+1}{10^{m!}}$$

where both $p = 10^{m!-1}+10^{m!-2}+\ldots+1$ and $q = 10^{m!}$ are integers. Using induction it's easily shown that for $r \ge 1$ $(m+r)! \ge (m+1)! + (r-1)$ so $\frac{1}{10^{(m+r)!}} \le \frac{1}{10^{(m+1)!+(r-1)}}$. Thus

$$\left|{}^{p}\!/_{q} - l_0\right| = \sum_{k=m+1}^{\infty}\frac{1}{10^{k!}} \le \frac{1}{10^{(m+1)!}} + \frac{1}{10^{(m+1)!}\cdot 10} + .. = \frac{1}{10^{(m+1)!}}\sum_{k=1}^{\infty}\frac{1}{10^k} = \frac{1}{10^{(m+1)!}}\left(\frac{10}{9}\right)$$

From Liouville's Inequality where $q = 10^{m!}$

$$\frac{1}{A} \le \left(10^{m!}\right)^n \left|{}^{p}\!/_{q} - l_0\right| \le \left(10^{m!}\right)^n \frac{1}{10^{(m+1)!}}\cdot\frac{10}{9} = \frac{1}{10^{(m+1)!-n\cdot m!}}\cdot\frac{10}{9} = \frac{1}{10^{m!\cdot(m+1-n)}}\cdot\frac{10}{9} \le \frac{1}{10^{m!}}\cdot\frac{10}{9}.$$

Since A is fixed we can make $m!$ arbitrarily large so that $\frac{1}{A} \ge \frac{1}{10^{m!}}\cdot\frac{10}{9}$ a contradiction!

As mentioned above, proving a number is transcendental is difficult. Hermite proved (1873) that e was transcendental (something Liouville set out to do but failed) and Lindemann proved (1882) the same for π, but it's still not known if certain numbers are transcendental or not. However, fallout from Cantor's 1874 result proved that the transcendental numbers are more numerous than the algebraic numbers though the former are harder to find. That's a funny asymmetry!

This brings to mind T.E. Bell's quote "The algebraic numbers are spotted over the plane like the stars against a black sky; the dense blackness is the firmament of the transcendentals"

We continue with another function similar to the one we started with. Given an algebraic number *a*, define $H_0(a)$ to be the height of the minimal degree polynomial $p_n(x)$ with integer coefficients such that $p_n(a) = 0$.

We now claim the function

$$h(x) = \begin{cases} 1/H_0(x) & \text{if } x \text{ is algebraic} \\ 0 & \text{if } x \text{ is transcendental} \end{cases}$$

is continuous for all transcendental values of x and discontinuous for all algebraic values of x. The proof is similar (?) to the one given above.

From the 20th century there is one further "that's funny" asymmetry found in the set of real numbers.

A. Turing
1912-1954

Define a computable number to be one whose digits can be generated by a Turing Machine even if the decimal expansion (as in the case of π) does not terminate; that is, there is a mechanical method to generate that number to any degree of accuracy. We note here that Turing's Thesis states that any computation that can be performed by some mechanical means can be done by a Turning machine which we can interpret that for any program run on a computer, there is a Turing machine that will do the same thing (though much slower).

The number of Turing Machine machines is countably infinite (there is a method to encode them as strings of binary integers) so the set of *computable numbers* is countably infinite. Thus the remaining set of non-computable numbers is non-denumerable and therefore in some sense non-obtainable? Another "that's funny" asymmetry?

To paraphrase T.E. Bell: The computable numbers are spotted over the plane like stars against a black sky of numbers which we can never obtain.

References

Dunham, William; The Calculus Gallery: Masterpieces from Newton to Lebesque; Princeton University Press, 2005.

Dunham, William; Journey through Genius; © John Wiley and Sons 1990; Penguin Books, New York, New York; 1991.

Grabiner, Judith; Who Gave You the Epsilon? Cauchy and the Originals of Rigorous Calculus; The American Mathematical Monthly, Vol. 90, No. 3. (Mar., 1983), pp 185-194

Kleiner, Israel; Evolution of the Function Concept: A Brief Survey; The College Mathematics Journal, Vol 20. No. 4 (Sept. 1989) pp. 282-300.

Linz, Peter; Formal Languages and Automata 6th Ed. Jones and Bartlett; Burlington MA; 2017

MacTutor History of Mathematics archives (May 2019). http://www-history.mcs.st-andrews.ac.uk/history/index.html