Finding the Volume of an Origami Frustum

Brian Shelburne
Professor Emeritus
Department of Mathematics and Computer Science
Wittenberg University

An Origami Frustum (see Figure 5), a truncated square pyramid with an open top, is easily constructed using a square sheet of paper (with sides of length s).

To start make two diagonal folds: AC and BD crossing at center O (Figure 1).

Next pinch the four side midpoints together and flatten out to form a 45-45-90 right triangle whose hypotenuse is square side (Figure 2).

Assuming the DC edge is in front, bend the D flap over so that the tip touches the opposite edge and is parallel to the top of the triangle. *Note that Δ AEF is congruent to Δ DFE by ASA and both are isosceles triangles* (Figure 3). As will be seen, this is critical to constructing the base *b* of the frustrum and the open top *a*.

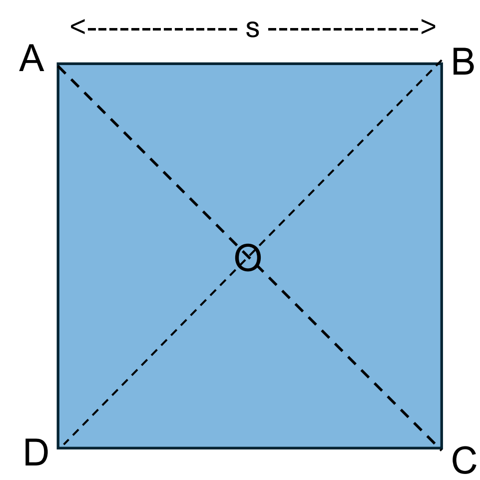

Figure 1

Do the same with the C flap (Δ CGD) but tuck its point into the envelope formed by the D flap triangle. (Note that C is tucked inside the envelope.). Note that lines AE = FD = BG

Turnover and repeat with the back-flaps forming a pentagonal-shaped flattened frustrum with a hole (EG) on the top and base FD (Figure 4).

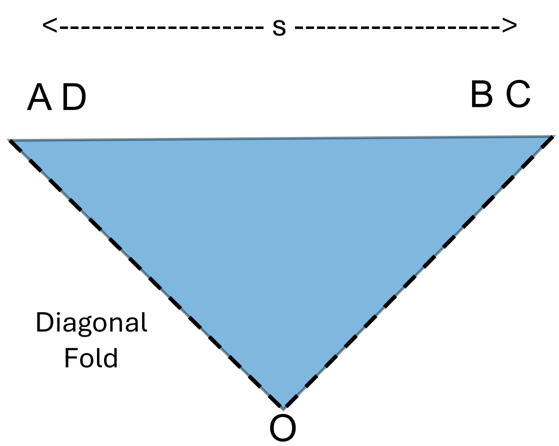

Figure 2

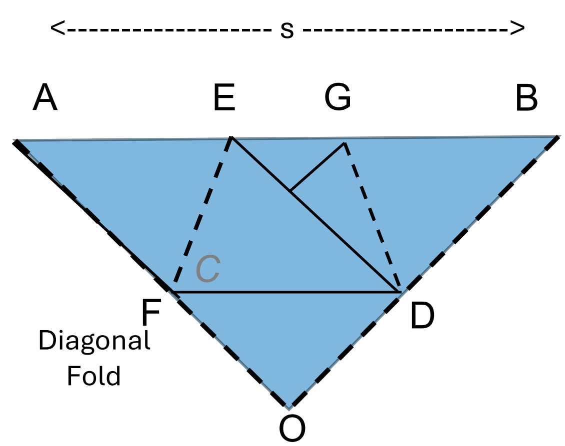

Figure 3

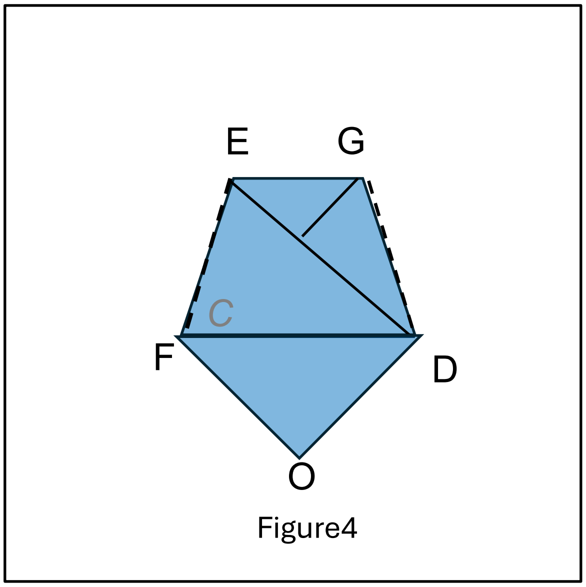

Figure4

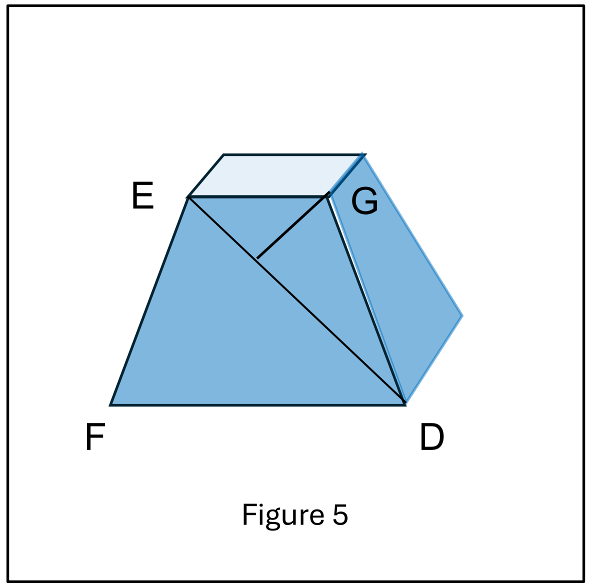

Figure 5

Finally expand the flattened pentagonal into the three-dimensional frustum (Figure 5) flattening the FOD triangle to form the base. (It helps to blow into the opening at the top of the figure!).

The volume for a frustum is given by the equation $V = \frac{1}{3}\left(a^2 + a \cdot b + b^2\right)$. To determine the volume of the origami frustum, we need to find values for the height *h*, and bottom base *b* and the top hole *a* given *s* the length of the side of the square paper. This is easily done (?) using some right triangle geometry.

To obtain values for *a, b* and height *h*, observe from Figure 6 that there are two 45-45-90 right triangles: ABO with sides $s, \frac{s}{\sqrt{2}}, \frac{s}{\sqrt{2}}$ and altitude (height) $\frac{s}{2}$ and FDO with sides $b, \frac{b}{\sqrt{2}}, \frac{b}{\sqrt{2}}$ and altitude (height) $\frac{b}{2}$.

Also recall from our construction that

AE = AF = FD = GB = BD = *b*

Thus AE + EG + GB = b + a + b = s and AF + FO = $b + \frac{b}{\sqrt{2}}$ = s. Solving these two equations for b then a yields

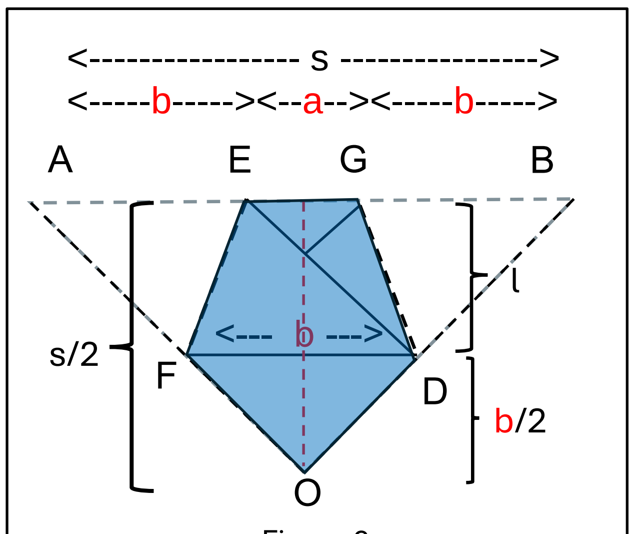

Figure 6

$$b = \frac{s}{1+\sqrt{2}} \text{ and } a = \left(3 - 2\sqrt{2}\right) \cdot s$$

In addition, from Figure 6, $l+\frac{b}{2}=\frac{s}{2}$ or $l=\frac{s-b}{2}$

Unfortunately, finding the height h is more complicated; we'll need to look at the frustum as a three-dimensional figure to determine h.

From Figure 7 the height of the frustum is the altitude or height of the red right triangle positioned in the three-dimensional frustum whose base is $\frac{b-a}{2}$ with hypotenuse l from Figure 6. Hence

$$h=\frac{\sqrt{(s-b)^2-(b-a)^2}}{2}=$$

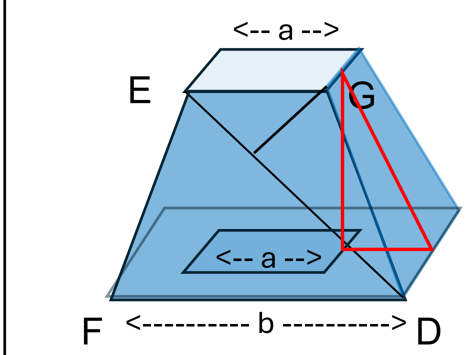

Figure 7

Thus, starting with s, the length of a side of the square used to construct the origami frustum, values for a, b, and h are found as functions of s.

Example: Given a 6 by 6 in square sheet of paper, what is the volume of the corresponding origami frustum? Using a TI-84 calculator (storing the values for a, b, and h)

$$a=\left(3-2\sqrt{2}\right)\cdot 6=1.029437252 \qquad b=\frac{6}{1+\sqrt{2}}=2.485281374$$

$$h=\frac{\sqrt{(6-b)^2-(b-a)^2}}{2}=1.599512809 \qquad V=\frac{h}{3}\left(a^2+ab+b^2\right)=5.222305776$$

Addendum – Deriving the formula for the volume of a frustum.

A (or the) straightforward approach to obtaining the volume of a frustum is to see it as the *remaining volume* of a pyramid with a smaller proportional pyramid cut off from the top.

That is from Figure 8 …

$$V_{frustum}=\frac{h_b}{3}b^2-\frac{h_b-h}{3}a^2$$

where we use $h_a=h_b-h$. Since the two pyramids are proportional, we have $\frac{h_b}{b}=\frac{h_b-h}{a}$ or

$$h\cdot b=h_b\left(b-a\right).$$

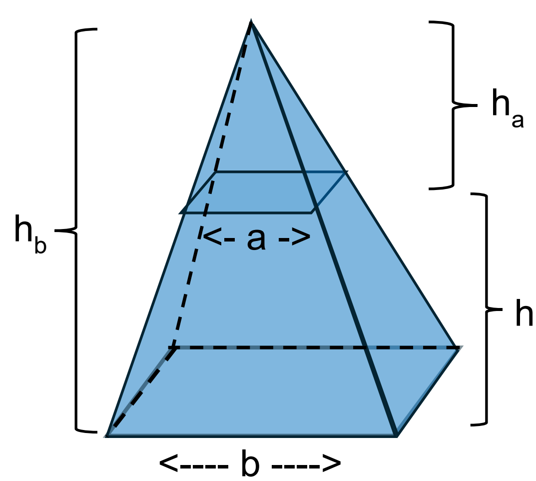

Figure 8

Therefore

$$V_{frustum} = \frac{h_b}{3}b^2 - \frac{h_b - h}{3}a^2$$
$$= \frac{h_b}{3}\left(b^2 - a^2\right) + \frac{h}{3}a^2$$
$$= \frac{1}{3}\left(h_b\left(b - a\right)\left(b + a\right) + ha^2\right)$$
$$= \frac{1}{3}\left(\left(h \cdot b\right)\left(b + a\right) + ha^2\right)$$
$$= \frac{h}{3}\left(b^2 + ab + a^2\right)$$

Note that the use of algebraic notation (3800 years after the date of the Moscow Papyrus) clearly shows how the formula for the volume of a frustum can be *easily derived* as the remaining volume of a pyramid with a smaller proportional pyramid cut off from the top. As noted above it's not clear how the ancient Egyptians did it – but they did!

Note: In his paper "The volume of a truncated pyramid in ancient Egyptian papyri" by R. J. Gillians *The Mathematics Teacher*, December 1964, Vol. 57 No 8 (December 1964), pp. 553 – 553, Gillians states

> "While it has been generally accepted that the Egyptians were well acquainted with the formula for the volume of the complete square pyramid, $V = \frac{h}{3}a^2$ it has not been easy to establish how they were able to deduce the formula for the truncated pyramid, *with the mathematics at their disposal, in its most elegant and far from obvious form* (italics mine)….

Gillians goes on and suggests a possible approach to how the Egyptians *might* have obtained the formula given in the Moscow Papyrus.

rev 05/06/24