1. 2. The 19th century saw an emphasis on mathematical rigor with mathematicians like Cauchy and Weierstrass rigorously defining mathematical concepts like limit, continuity, etc. which resulted in a flood of pathological (?) functions testing these definitions.

3. Out of this milieu comes one of my favorite "that's funny" result in mathematics: the existence of a function on the domain (0,1) which is continuous at all irrational points and discontinuous at all rational points (while no function seems to exist which is the other way around):

$$f(x)=\begin{cases} 1/q \ \ if\ x = p/q \quad \gcd(p,q)=1 \\ 0 \ \ if\ x\, is\, irrational \end{cases}$$

Showing continuity at irrational points is a nicely done using a straight-forward δ –ε proof - demonstrating the power of the *modern* definition for limit and continuity. Kudos to Cauchy and Weierstrass!

The δ-ε proof of continuity at an irrational number x_0 uses the fact that for any ε > 0 and integer r where 1/r < ε there are finitely many rational numbers p/q where q ≤ r . Let δ > 0 be the minimal distance between x_0 and this finite set of rational numbers. If q > r then |*f*(p/q) – 0|< 1/q < 1/r < ε. In some sense irrationals x_0 avoid rational numbers p/q with denominators greater than some integer r.

4. In 1874 Cantor proved the real numbers are non-denumerable. Given that the rational numbers are countably infinite and the union of two countably infinite sets is countably infinite, assuming the irrational numbers are countably infinite contradicts Cantor's result. In some sense there are more irrationals than rational numbers.

A superset of the rational numbers, the algebraic numbers, which are the roots of polynomials with integer coefficients is also countably infinite, while its set complement, the transcendental numbers, is non-denumerable. The infinite countability of the algebraic numbers is based on the height of a polynomial with integer coefficients. Specifically

$$H\left(a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_o\right) = \sum_{k=0}^{n} |a_k| + n - 1$$

5. Since the set of polynomials with integer coefficients with a fixed height is finite, it's not difficult to show the algebraic numbers are also countably infinite. What makes this asymmetry interesting is the difficulty in finding a transcendental number. In 1851 Liouville proved the number

$$6.\quad l_0 = \sum_{k=0}^{\infty} \frac{1}{10^{k!}}$$

was transcendental using an inequality necessary for irrational algebraic numbers.

7. If x_0 is an irrational algebraic number with minimal degree polynomial $p_n(x)$ having integer coefficients, then there is a positive number A such that if p/q is a rational number in the interval $[x_0-1, x_0+1]$ then

$$\left| p/q - x_0 \right| \geq \frac{1}{Aq^n}$$

That is, irrational algebraic numbers in some sense avoid rational numbers similar (?) to irrational numbers avoiding (?) certain rational numbers (mentioned above). And if for any algebraic number a, we define $H_0(a)$ to be the height of the minimal degree polynomial with integer coefficients for which a is a root, then the function

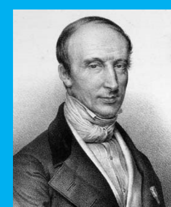
$$h(x)=\begin{cases} 1/H_0(x) \ \ if\ \ x\, is\, algebraic \\ 0\ if\ x\ is\ transcendental \end{cases}$$

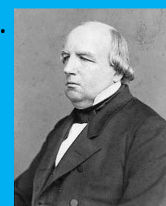
is continuous for all transcendental values of x and discontinuous for all algebraic values of x using a proof is similar (?) to the one given above. **5.**

Exploring Asymmetrical Results in Mathematics

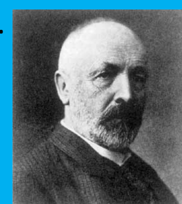
Set Asymmetries in the Real Numbers

Brian Shelburne - Wittenberg University

1.


2.


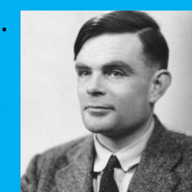
3.
$$f(x)=\begin{cases} 1/q \ \ if\ x = p/q \\ 0\ if\ x\, is\, irrational \end{cases}$$

4.


5.


6.
$$l_0 = \sum_{k=0}^{\infty} \frac{1}{10^{k!}}$$

7.
$$\left| p/q - x_0 \right| \geq \frac{1}{Aq^n}$$

8.


The most exciting phrase to hear in ~~science~~ mathematics – the one that heralds new discoveries is not "Eureka" but "That's funny…"

Isaac Asimov with apologies to Archimedes

8. A superset of the algebraic numbers are the computable numbers, numbers whose digits can be generated by a Turing Machine even if the decimal expansion (as in the case of π) does not terminate. Turing's Thesis states that any computation that can be performed by some mechanical means (i.e. a computer) can be done by a Turning machine. Since number of Turing Machine machines is countably infinite (there is a method to encode them as strings of binary integers), the set of computable numbers is countably infinite. Thus the remaining set of non-computable numbers is non-denumerable and therefore in some sense non-obtainable (?).