Math Monday

August 30, 2010

A Mathematical Potpourri: Conundrums, Puzzles, Tricks and Other Trivia!

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Let's begin with Gauss (1777 – 1855)

Did he do it this way?

$$1+ \quad 2+ \quad 3+...+ \ 99+100$$
$$100+ \ 99+ \ 98+...+ \quad 2+ \quad 1$$
$$_____$$
$$101+101+101+...+101+101$$

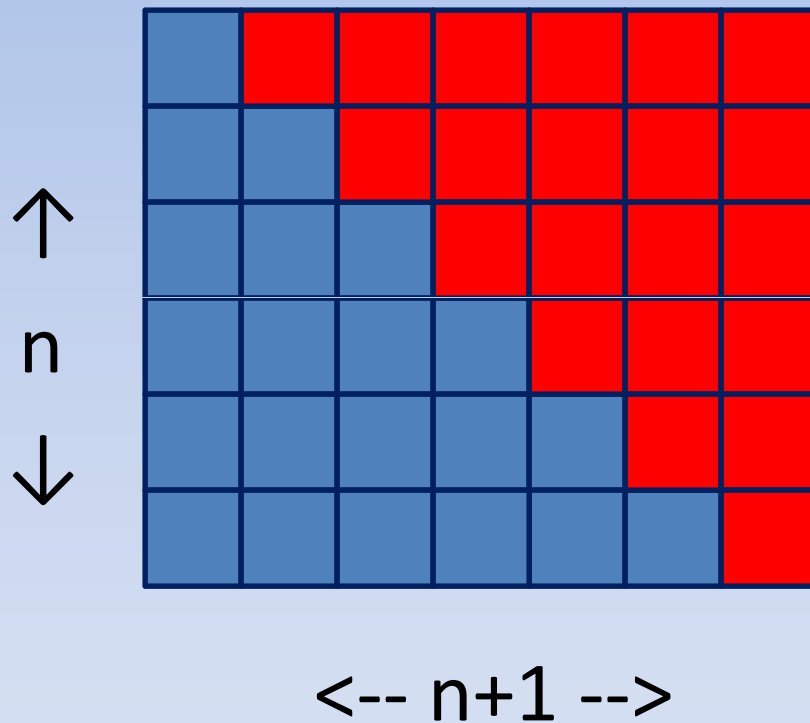obtaining 100 copies of 101 (twice the sum) so that the sum is

$$\frac{100\times101}{2}=5050$$

In any case we have this very nice formula

$$\sum_{k=1}^{n} k = 1 + 2 + 3 + \ldots + n = \frac{n \cdot (n+1)}{2}$$

which is easy to remember especially ….
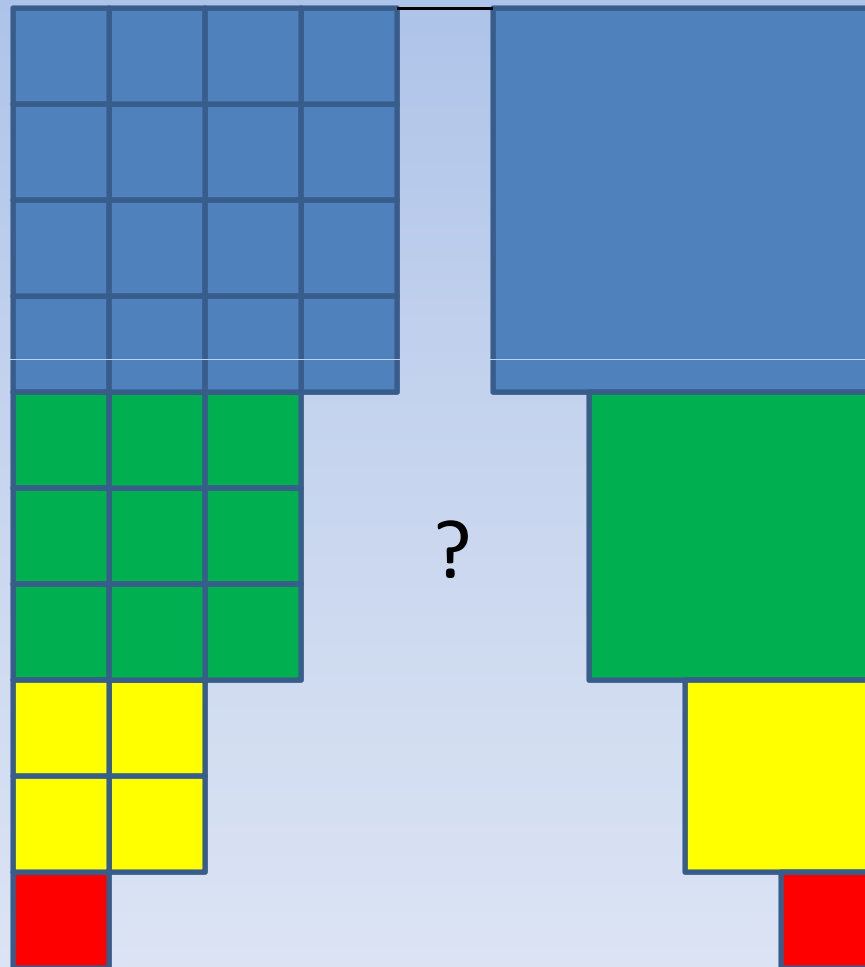
Proof without Words I

↑

n

↓

<-- n+1 -->

two 1+2+3+..+n

triangles form

an n by (n+1)

rectangle

But what about the sum of squares; that is
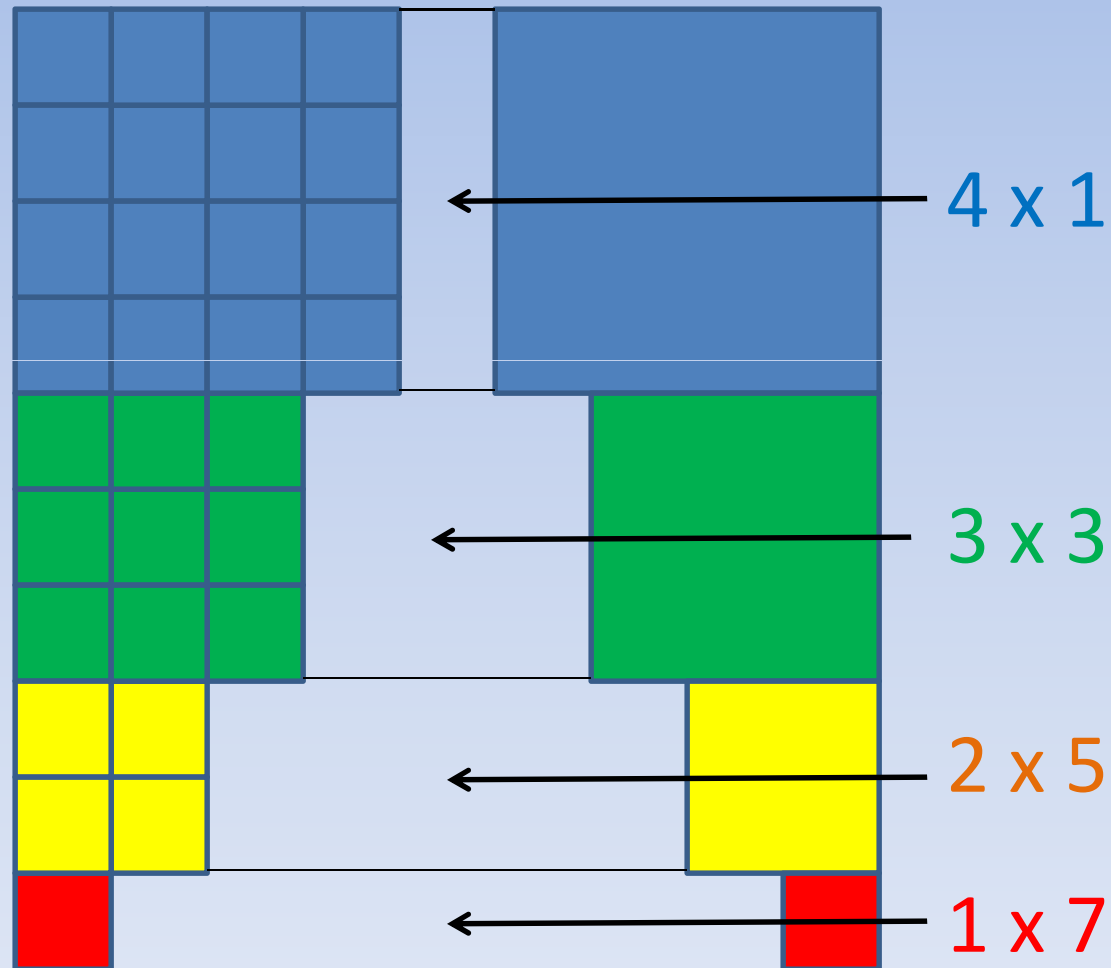
$$1 + 4 + 9 + 16 + \ldots + n^2$$

Proof without Words II

Observe: 3 (?) copies of $1+4+9+...+n^2$

Proof without Words II

Observe: 3 copies of $1+4+9+...+n^2$

4 x 1

3 x 3

2 x 5

1 x 7

There's a pattern

consecutive odd integers

1 x 7 = 7

2 x 5 = 5 + 5

3 x 3 = 3 + 3 + 3

4 x 1 = 1 + 1 + 1 + 1
_____

= 16 + 9 + 4 + 1

Proof without Words III

n^2 is the sum of the first n odd integers

Proof without Words II

Observe: a (1+2+3+…+n) x (2n+1) rectangle

$$\sum_{k=1}^{n} k$$

4 x 1

3 x 3

2 x 5

1 x 7

1+2+3+…+n

<----------- 2n+1 ----------->
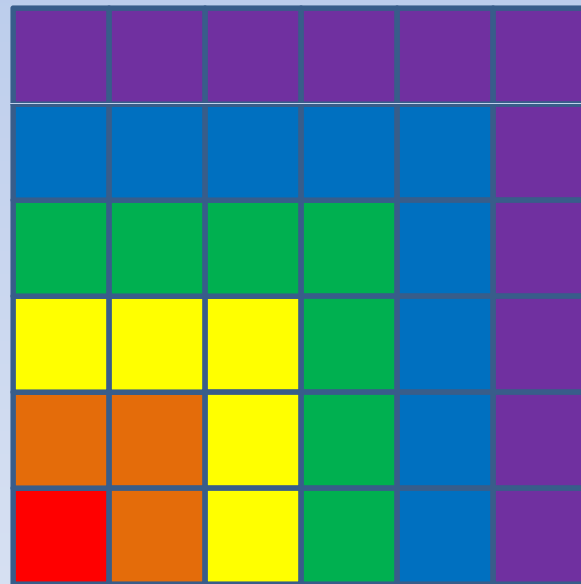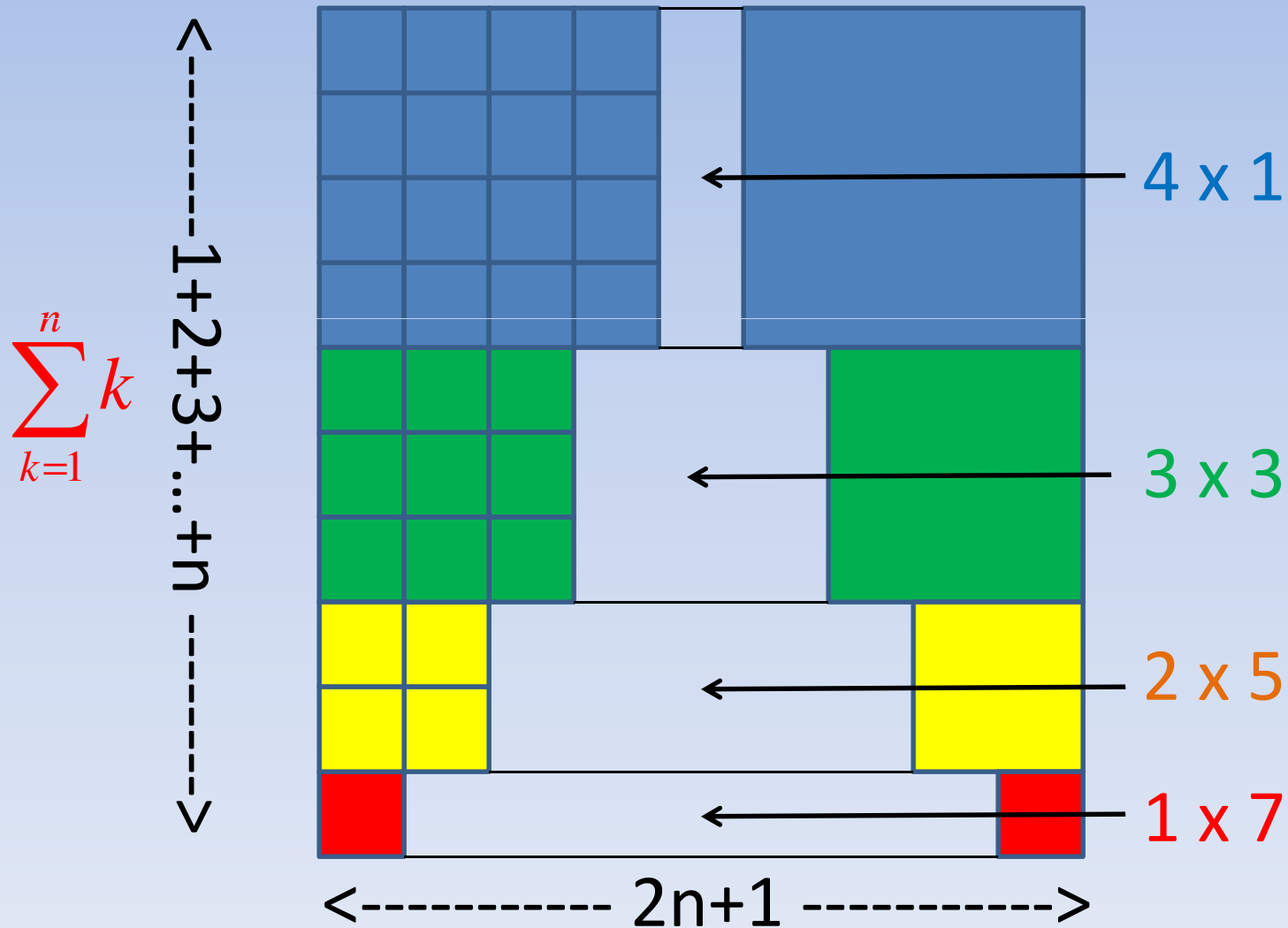
Therefore $\displaystyle\sum_{k=1}^{n} k^2 = 1 + 4 + 9 + \ldots + n^2 =$

$$\frac{\displaystyle\sum_{k=1}^{n} k \times (2n+1)}{3} =$$

$$\frac{n(n+1)(2n+1)}{2 \cdot 3} = \frac{n(n+1)(2n+1)}{6}$$

And that's how I remember the formula!

A Card Trick Guessing Game

- Pick a number between 1 and 31

- There are 5 cards with numbers – tell me which cards (i.e. colors) have this number

- I will tell you the number!

The Cards

RED			
1	3	5	7
9	11	13	15
17	19	21	23
25	27	29	31

ORANGE			
2	3	6	7
10	11	14	15
18	19	22	23
26	27	30	31

GREEN			
4	5	6	7
12	13	14	15
20	21	22	23
28	29	30	31

BLUE			
8	9	10	11
12	13	14	15
24	25	26	27
28	29	30	31

and the last card

VIOLET			
16	17	18	19
20	21	22	23
24	25	26	27
28	29	30	31

The Card Trick Revealed

- Each card (color) is a different "amount"

  Red is 1

  Orange is 2

  Green is 4

  Blue is 8

  Violet is 16

- For every color chosen add that amount! The sum is the secret number!

The Card Trick Revealed - Binary Numbers

Every integer can be uniquely expressed a sum of the powers of 2

Example: $5 = 4 + 1 = 2^2 + 2^0 = 101_2$

$11 = 8 + 2 + 1 = 2^3 + 2^1 + 2^0 = 1011_2$

$28 = 16 + 8 + 4 = 2^4 + 2^3 + 2^2 = 11100_2$

So the colors chosen select the binary representations of the integers between 1 and 31

Car Talk Puzzler – pt 1

RAY: This was from my 'revolt of the lab rats' series and it was sent in interestingly enough by a fellow named Andrew Magliozzi, who happens to be my son! I thought the name looked familiar

Imagine this: It's the eve of the annual Car Talk 'We Haven't Been Canceled Yet' Banquet. John 'Bugsy' Lawlor has procured the food and the wine for the following evening's gala, and he's extremely pleased with himself because he knows a guy, who knows a guy, who knows a guy, and he's obtained for $50, thirteen one-half gallon jugs of the finest red wine. He's very excited about the following evening's festivities until Doug 'Punkin Lips' Berman informs him that he received a note suggesting that one of those 13 bottles of wine contains a deadly poison. In fact, it's one that kills within 24 hours. 'Is someone out to get us? Who could be out to get us? Why? Why not? But why the whole staff?'

Car Talk Puzzler – pt 2

Anyway, because they're hopeless cheapskates and unwilling to discard 12 bottles of perfectly good cheap wine, they head off to MIT to consult with a scientist friend of theirs. 'So let's see,' he says, 'you have these thirteen bottles of wine and one of them contains a poison that will prove fatal within 24 hours after it's been consumed. And the Car Talk Extravaganza is when?'

'Tomorrow night.'

The scientist goes on to suggest that from his knowledge of poisons, even the smallest sample is usually enough to cause certain death, even if mixed and diluted with the wine from the untainted bottles. 'Hmm,' he says, 'I'll be right back.'

Car Talk Puzzler – pt 3

In a flash, he returns with four small cages, each one containing your standard lab rat. What a moral dilemma, they think, we have to sacrifice lab rats to save Car Talk? 'But wait, wait,' they say. 'We have 13 bottles of wine. How are we supposed to save the entire Car Talk staff and empire, with just 4 rats?'

'You can do it,' he says and then disappears into the inky shadows.

So, you have 4 rats in little cages, 13 bottles of wine and one of them has got poison in it. Now you can obviously take samples from any of the bottles and you can give as much or as little as you want to any of the rats. And don't forget, you're not going to know if the poison is fatal until the night of the banquet because it takes 24 hours to kill human or rat. So, how do you do it?
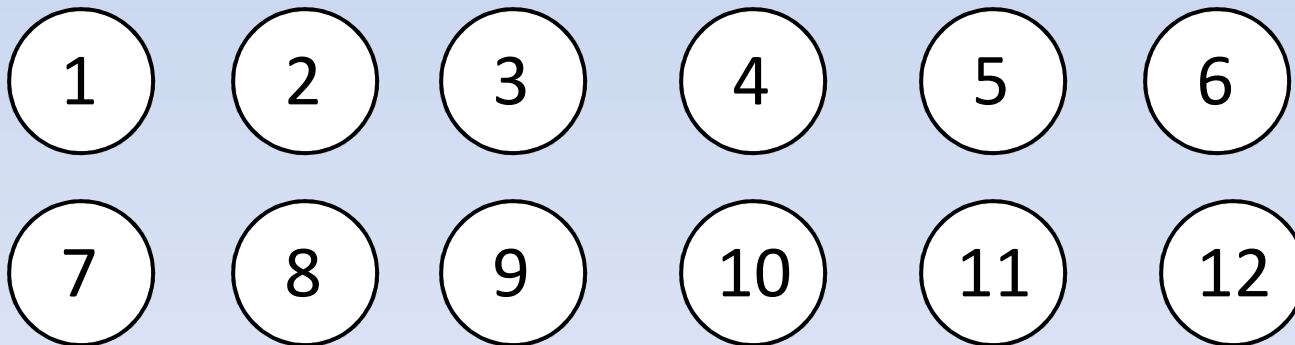
Car Talk Puzzler - Solution

Number the Rats 1, 2, 4, and 8 and feed them a mixture of the wine from the following bottles according to the following table.

Bottle	1	2	3	4	5	6	7	8	9	10	11	12	13
Rat #1	X		X		X		X		X		X		X
Rat #2		X	X			X	X			X	X		
Rat #4				X	X	X	X					X	X
Rat #8								X	X	X	X	X	X

The sum of the dead rats is the poisoned wine!

The Bogus Billiard Ball Problem

1 2 3 4 5 6

7 8 9 10 11 12

Bogus Billiard Ball Trick

1. You have 12 billiard balls numbered 1 -12 one of which is either heavier or lighter than the rest.

2. You have a two-pan balance with which to compare the weights of sets of balls.

3. Using the scale three times, find the bogus ball and tell if it's heavier or lighter than the rest.

Trial #1

Left	Right
1 4 7 8	2 5 10 11

Trial #2

Left	Right
2 3 4 11	5 6 7 12

Trial #3

Left	Right
5 6 7 11	8 9 10 12

Bogus Billiard Ball Puzzle

	Left				Right			
Trial 1:	1	4	7	8	2	5	10	11
Trial 2:	2	3	4	11	5	6	7	12
Trial 3:	5	6	7	11	8	9	10	12

If left side is heavier add +1 (trial 1), +3 (trial 2) or +9 (trial 3); else add respectively -1 , -3 or -9. If sum is + then heavier else lighter except reverse this for 9, 10, and 12

Ternary (Base 3) Representation

Similar to binary, ternary notation uses powers of 3 to represent numbers.

Example:

$$7 = 2 \times 3 + 1 = 2 \cdot 3^1 + 1 \cdot 3^0 = 21_3$$

$$17 = 1 \times 9 + 2 \times 3 + 2 = 1 \cdot 3^2 + 2 \cdot 3^1 + 2 \cdot 3^0 = 122_3$$

$$27 = 1 \cdot 3^3 = 1000_3$$

Observe that only the digits 0, 1, and 2 are used!

(How can you check if a number is divisible by 3?)

Balanced Ternary Notation

Base 3 except use digits +1, 0 and -1

Example :

$$7 = 2 \times 3 + 1 = 2 \cdot 3^1 + 1 \cdot 3^0 = 21_3$$

$$7 = +1 \times 9 + (-1) \times 3 + 1 = +1 \cdot 3^2 + (-1) \cdot 3^1 + 1 \cdot 3^0$$

Use +, 0 , and – for balanced ternary digits

$$17 = 1 \times 3^3 + (-1) \cdot 3^2 + (-1) \cdot 3^0 = +-0-_3$$

Balanced Ternary Notation is called the "Goldilocks" of Numbers

Bogus Billiard Ball Puzzle

	Left	Right
Trial 1:	1 4 7 8	2 5 10 11
Trial 2:	2 3 4 11	5 6 7 12
Trial 3:	5 6 7 11	8 9 10 12

If left side is heavier add +1 (trial 1), +3 (trial 2) or +9 (trial 3); else add respectively -1 , -3 or -9. If sum is + then heavier else lighter except reverse this for 9, 10, and 12

(Note: In order for the above to balance properly, we have to reverse the positions of balls 9, 10, and 12)

Question: Why do mathematicians confuse
   Halloween with Christmas

Answer: Because

$$31_{oct} = 25_{dec}$$

Common Fallacy I

Proof that 1 = 0: Let $x = 1$

$$x^2 = x$$

$$x^2 - 1 = x - 1$$

$$(x+1)(x-1) = x - 1$$

$$\frac{(x+1)(x-1)}{x-1} = \frac{x-1}{x-1}$$

$$x + 1 = 1$$

$$x = 0$$

Therefore 1 = 0!

Common Fallacy II
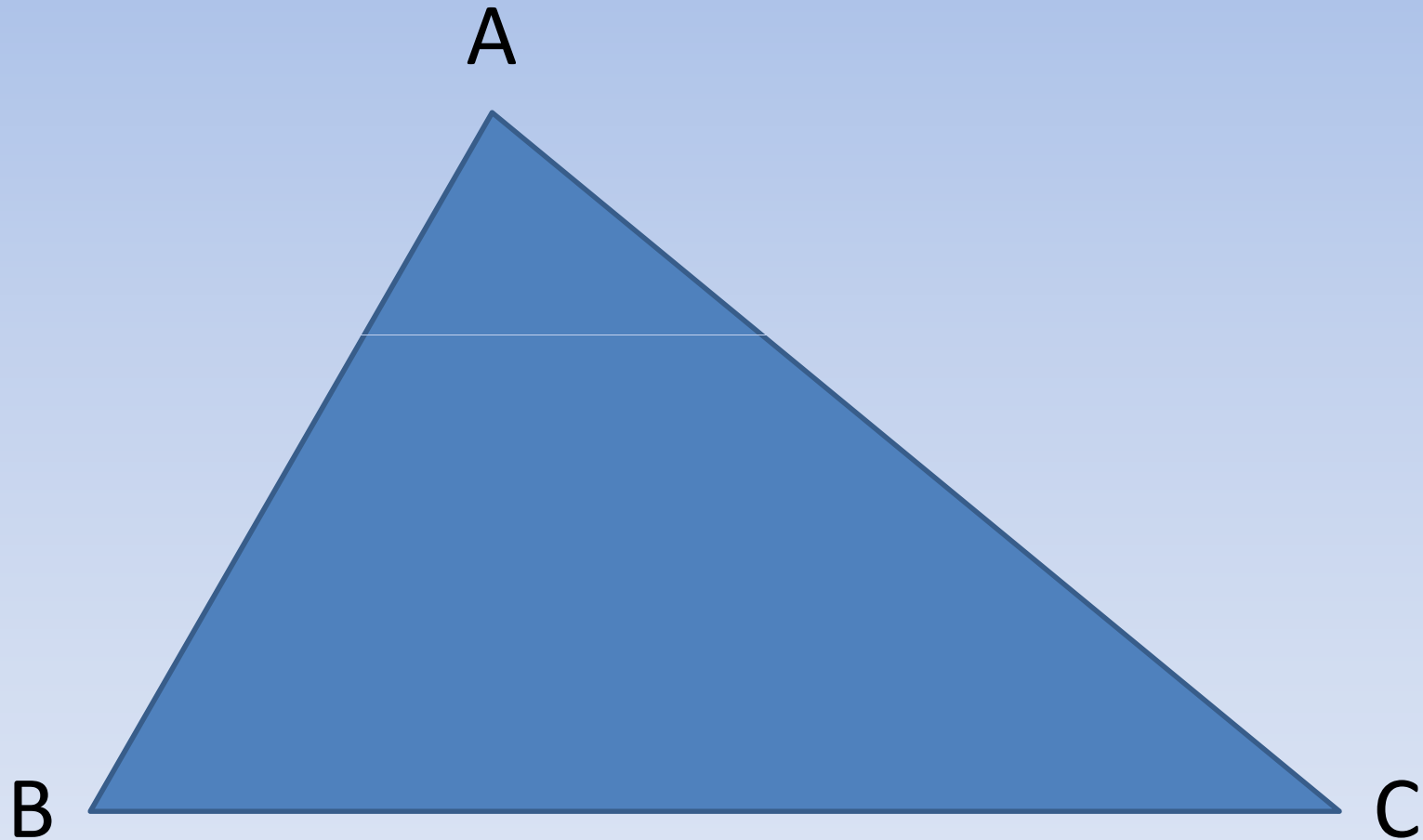
Proof that -1 = 1: $\quad -1 = \sqrt{-1} \cdot \sqrt{-1}$

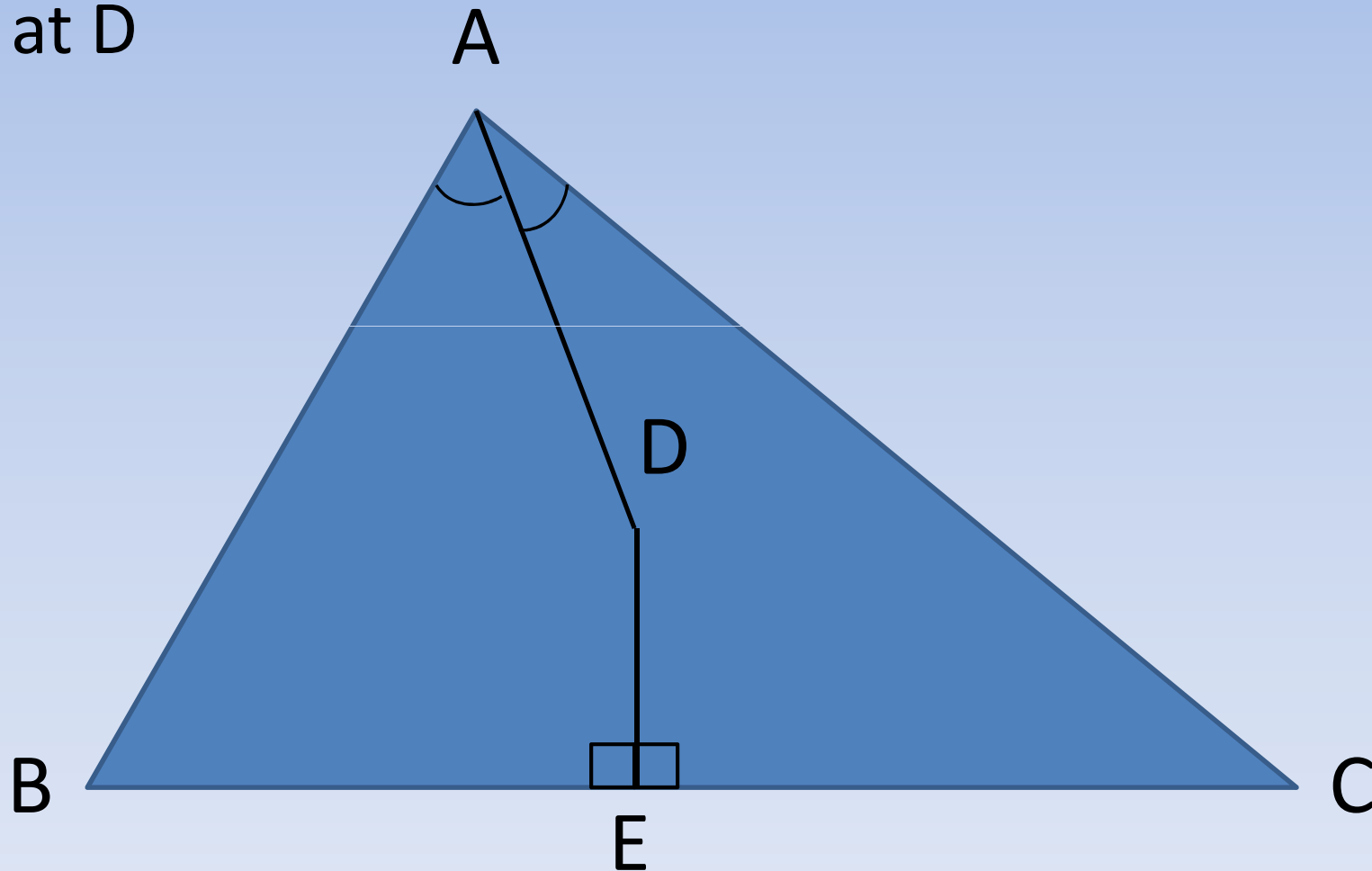$$\sqrt{-1} \cdot \sqrt{-1} = \sqrt{(-1)^2}$$

$$\sqrt{(-1)^2} = \sqrt{1} = 1$$

Therefore -1 = 1

Common Fallacy III

All triangles are isosceles.

Common Fallacy III

Let the bisector of $\angle A$ meet the $\perp$ bisector of $\overline{BC}$ at D
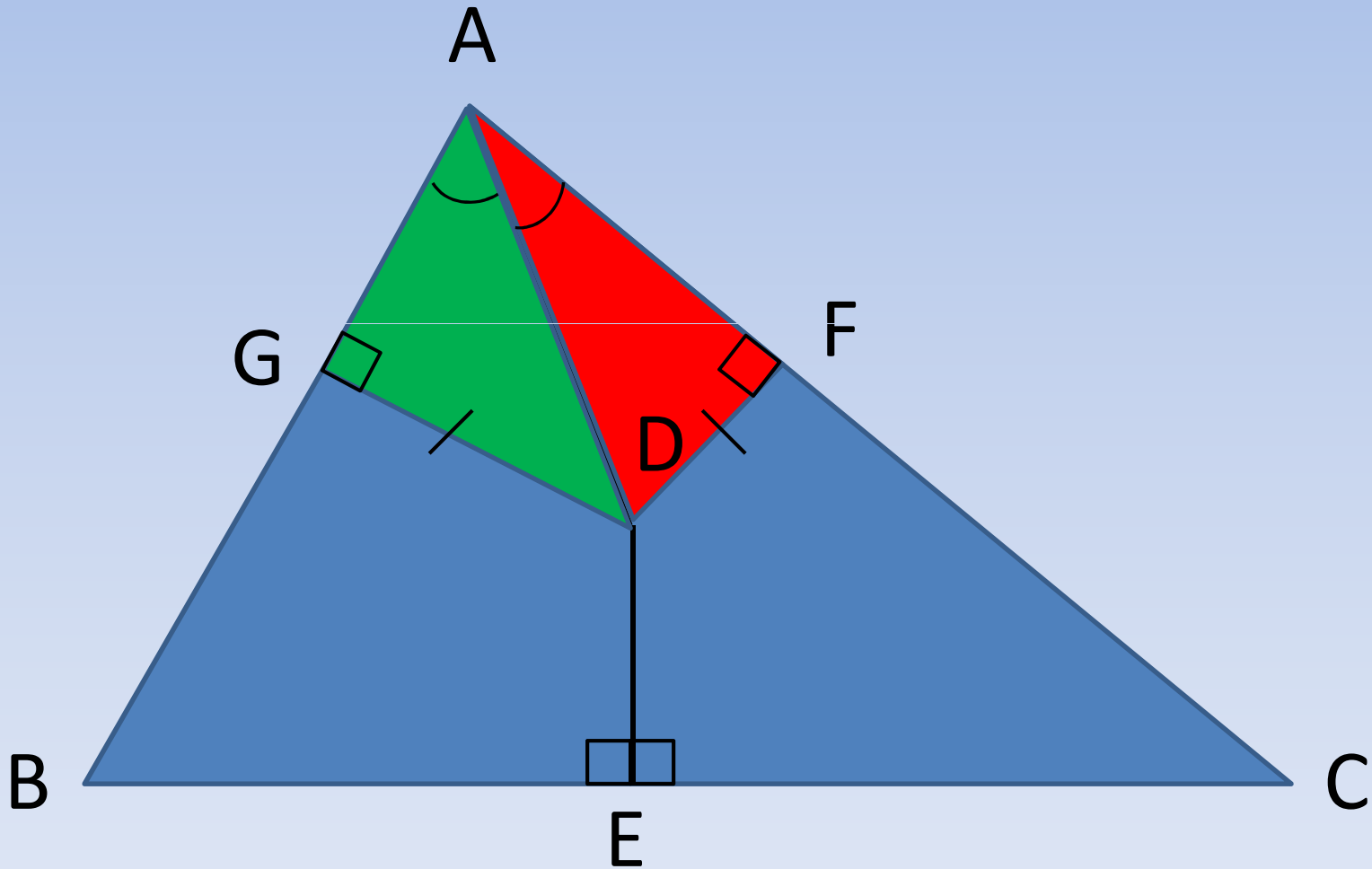
Common Fallacy III

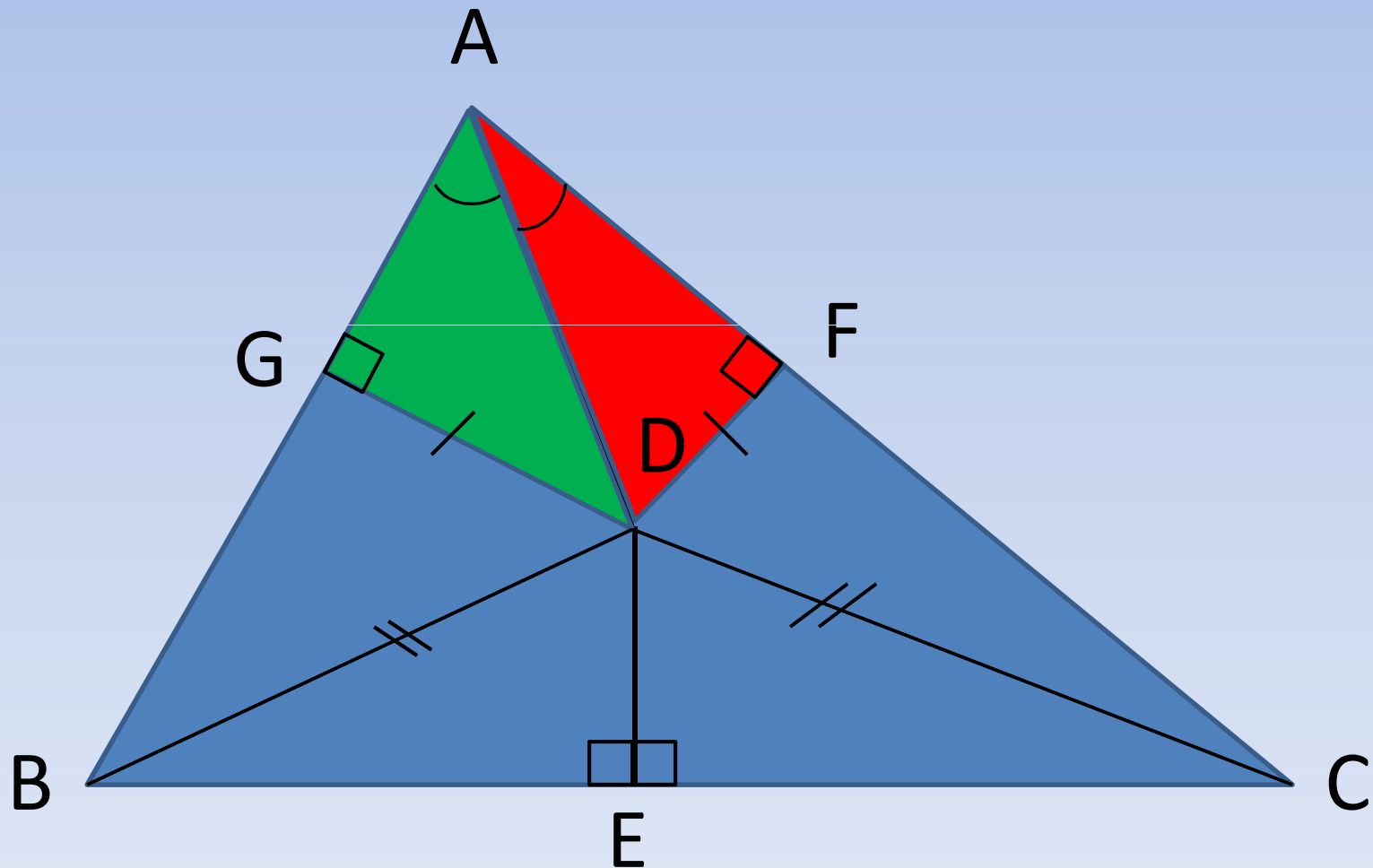Drop perpendicular lines from D to $\overline{AB}$ and $\overline{AC}$
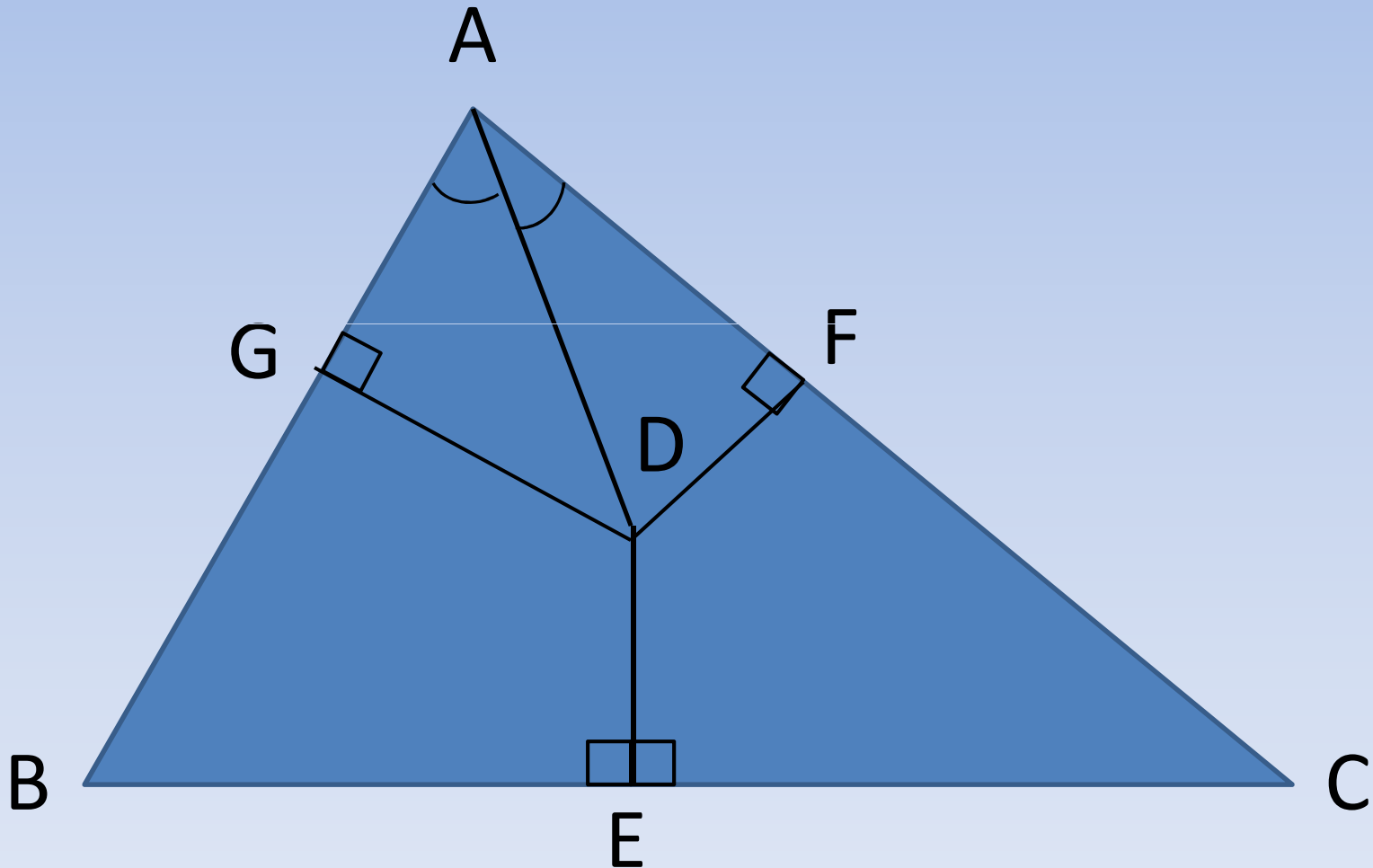
Common Fallacy III

It follows that $\Delta ADG \cong \Delta ADF$

Common Fallacy III

Since $\overline{DE}$ is a $\perp$ bisector $\triangle BED \cong \triangle CED$

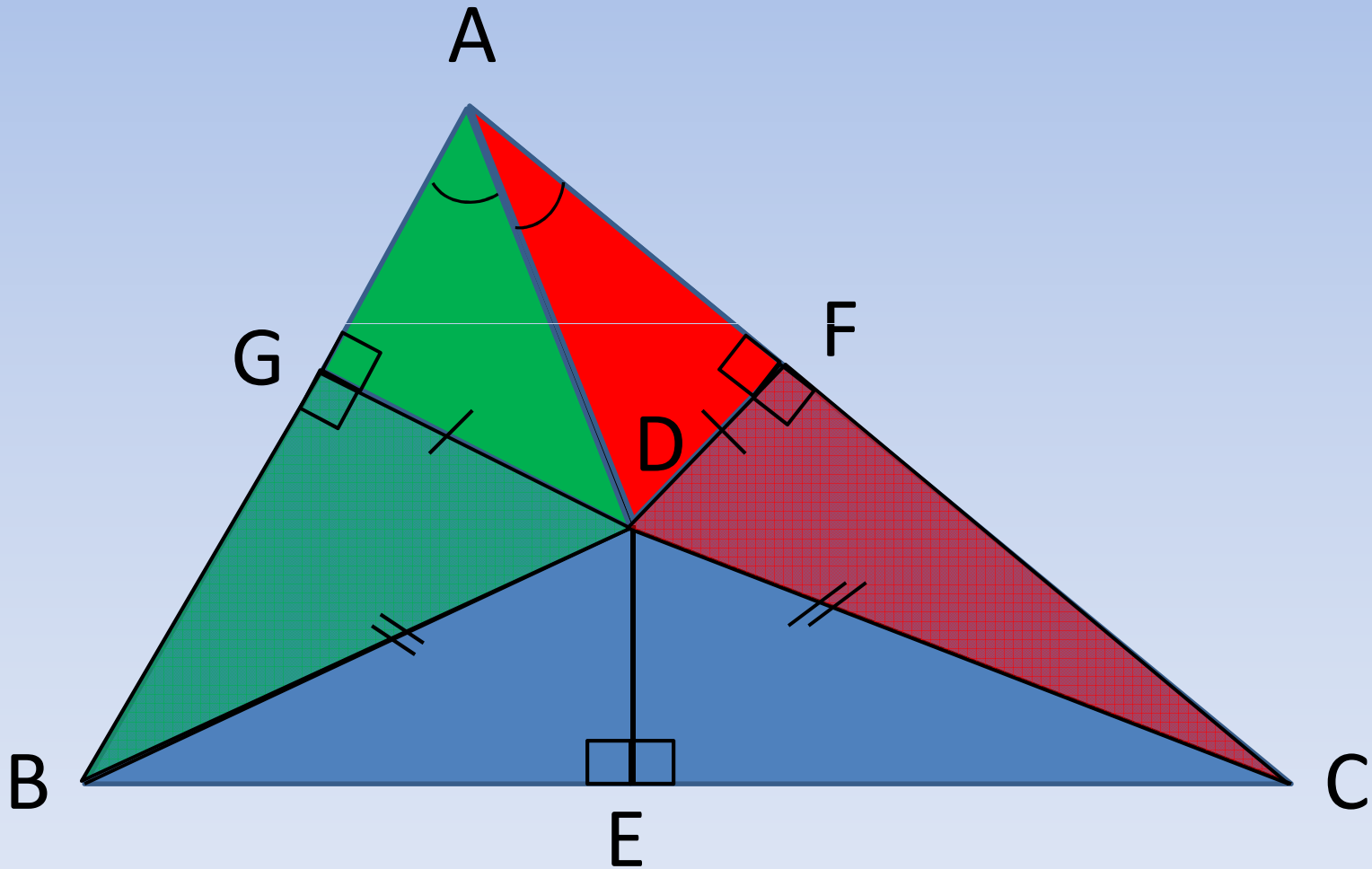Aside: If two right triangles have two equal sides then the two right triangles are congruent !
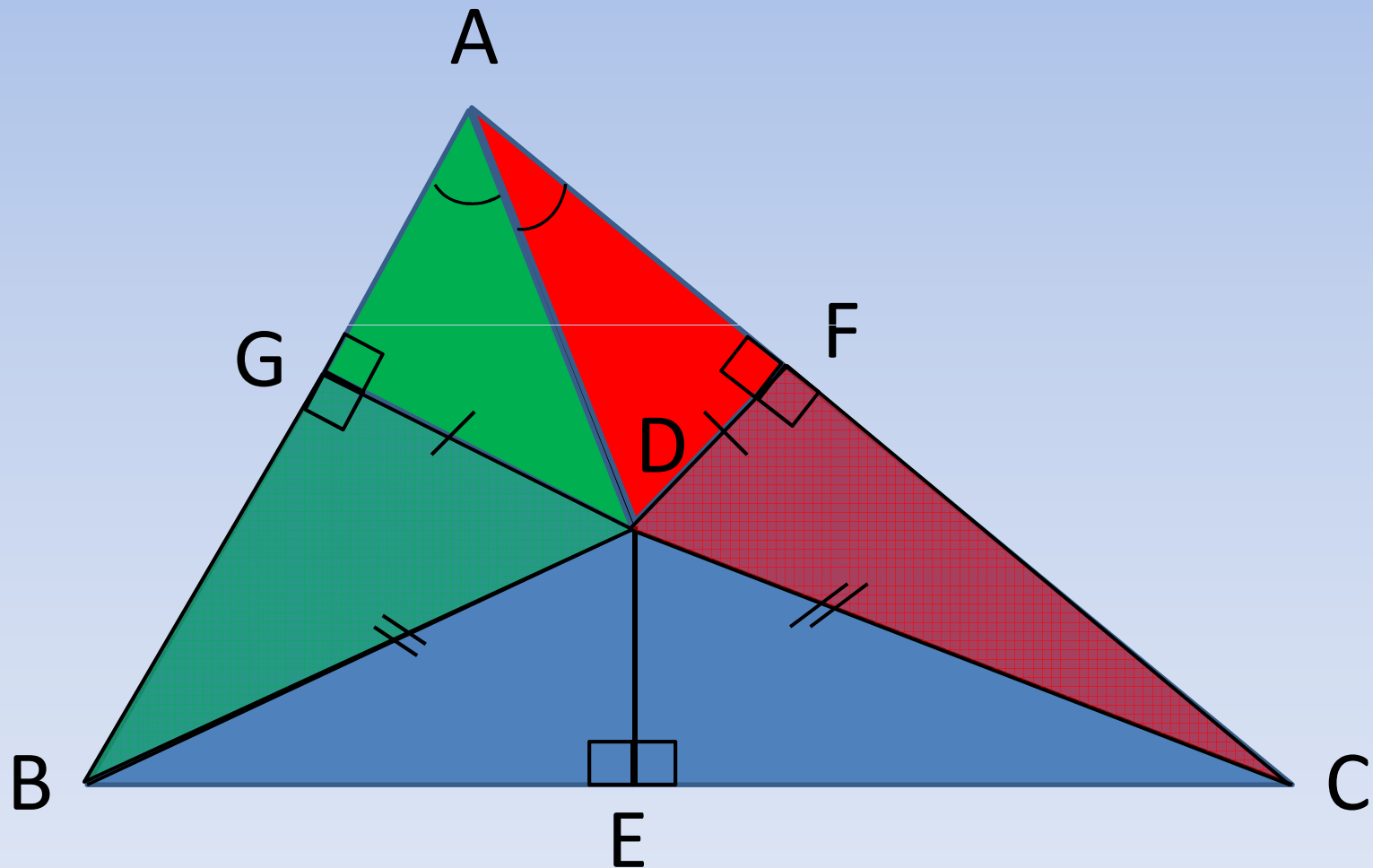
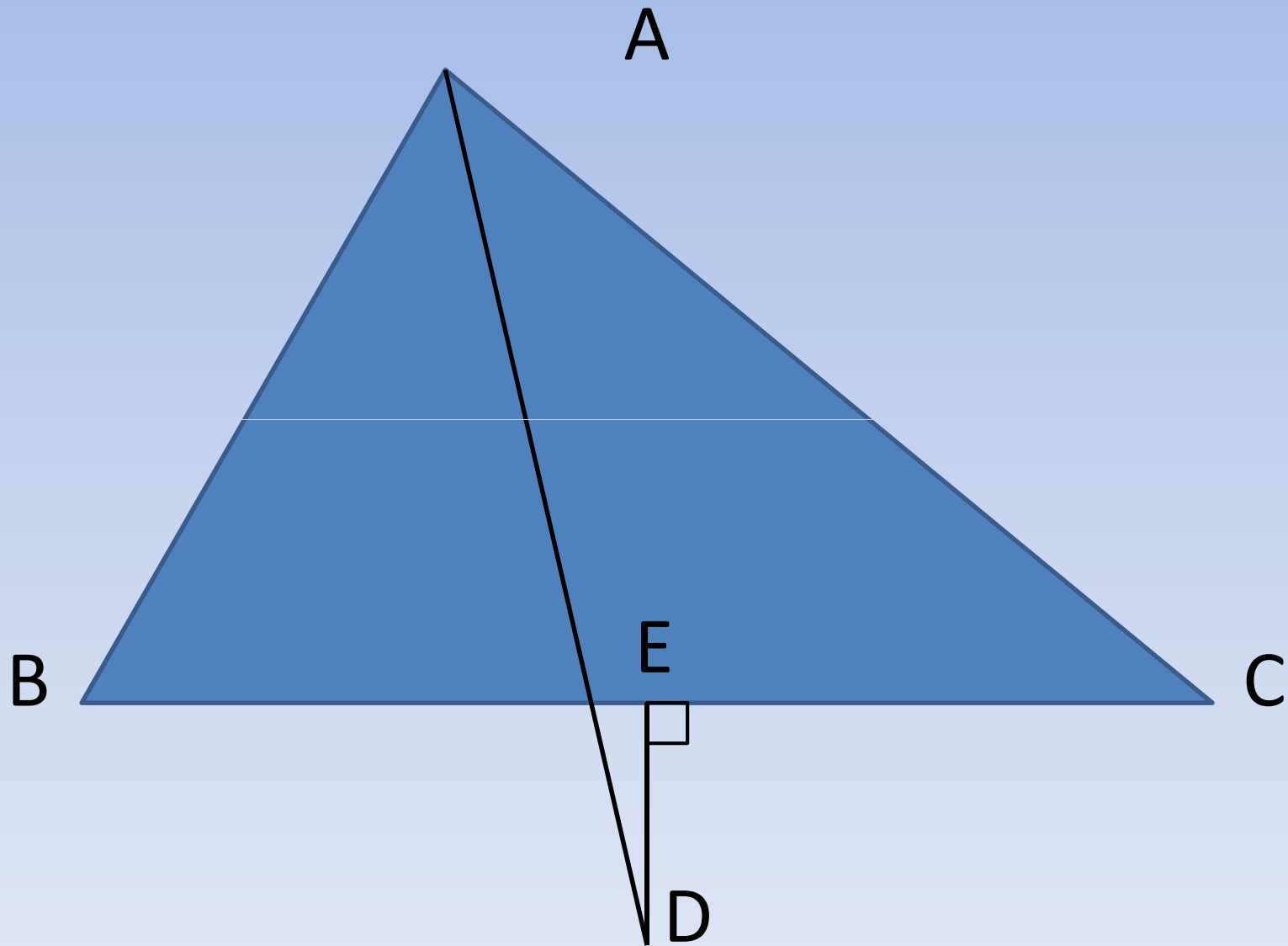Why?

Common Fallacy III

$\Delta BDG \cong \Delta CDF$

Common Fallacy III

Since $\overline{AG} = \overline{AF}$ and $\overline{GB} = \overline{FC}$, $\triangle ABC$ is isosceles

Common Fallacy III -Revealed

What so special about the number
05882352941117647?

Ok – how about 142857?

Hint: Try multiplying it by 2, 3, 4, …,16

The number 05882352941117647

0588235294117647
1176470588235294
1764705882352941
2352941176470588
2941176470588235
3529411764705882
4117647058823529
4705882352941176
5294117647058823
5882352941176470
6470588235294117
7058823529411764
7647058823529411
8235294117647058
8823529411764705
9411764705882352

Where did 0588235294117647 come from?

$$\frac{1}{17} = 0.\overline{0588235294117647}$$

Farey Sequences

True or False: $\dfrac{a}{b} \times \dfrac{c}{d} = \dfrac{a \times c}{b \times d}$

True or False: $\dfrac{a}{b} + \dfrac{c}{d} = \dfrac{a + c}{b + d}$

Farey Sequence

A Farey Sequence is the ordered (by magnitude) sequence of factions of the form a/b such that

1. $0 \le a/b \le 1$

2. $a < b$ and $b \le some\ constant\ c$

3. a/b are in lowest terms (so no "repeats")

Example $b \le 2: \dfrac{0}{1}, \dfrac{1}{2}, \dfrac{1}{1}$

Farey Sequences

Examples

$$b \le 3 : \frac{0}{1}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{1}{1}$$

$$b \le 4 : \frac{0}{1}, \frac{1}{4}, \frac{1}{3}, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{1}{1}$$

$$b \le 5 : \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

$$b \le 6 : \frac{0}{1}, \frac{1}{6}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{5}{6}, \frac{1}{1}$$

Farey Sequences

Observe for Farey Sequences

$$\frac{a}{b} + \frac{c}{d} = \frac{a+c}{b+d}$$

$$(b \le 5) \quad \frac{0}{1}, \frac{1}{5}, \frac{1}{4}, \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}, \frac{2}{3}, \frac{3}{4}, \frac{4}{5}, \frac{1}{1}$$

Observe that any entry is the "farey sum" of its neighbors

The Infinite Lottery

Set Up

1. Given: An infinite number of bags numbered 1, 2, 3, … each containing an infinite number of lottery balls.

2. The lottery balls in each bag are marked with the number on the bag (e.g. all bag 1 balls are marked with a "1" etc).

3. You have a box. Select any number (finite) of lottery balls from any number (finite) of bags and place them in the box

The Infinite Lottery

The Play

5. Remove any lottery ball from the box and replace it with any number (finite) of balls marked with a lower number.

6. Note: Removing a "1" lottery ball allows no replacements (why)

7. If you can devise a method to play the game forever – you win; however if the play terminates because the box is empty you lose.

Question: Do you win or lose?

The End

Next Math Monday will be Sept. 13 at 4 PM

Thank you!

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