Mathematical Riffs

The mathematics of the poetry found in

Manifold

poetry of mathematics
3: A Taos Press © 2021
by E.R. Lutken

"**If** Poetry uses *word play* to express the deeper realities of life **and if** mathematics uses *number play* to reveal the deeper realities in the universe of number, **then** it is not surprising that on some deep level the two have an intimate connection and the one can serve as a source for inspiration for the other."

0 1 2 3 4 5 6 7 8 9 10
11 12 13 14 15 16 17 18 19 20
21 22 23 24 25 26 27 28 29 30
31 32 33 34 35 36 37 38 39 40
41 42 43 44 45 46 47 48 49 50
51 52 53 54 55 56 57 58 59 60
61 62 63 64 65 66 67 68 69 70
71 72 73 74 75 76 77 78 79 80
81 82 83 84 85 86 87 88 89 90
91 92 93 94 95 96 97 98 99 100
101 102 103 . . . .

Brian J. Shelburne

Introduction

"Sire, there is no Royal Road to Geometry" – arr. Euclid

There are poems which have mathematical themes, poems where mathematics is the inspiration for a poem: thus, from mathematics can come poetry. But why not reverse the process – why not view a poem as an entrée into mathematics?

Manifold: poetry of mathematics (3: A Taos Press © 2021) by E R Lutken is a collection of 57 poems based on mathematics and/or mathematical themes. A subset of these poems, 27 in all, are the starting point for a series of 27 essays (*mathematical riffs*) based on that poem which explore and develop the mathematics, mathematical themes, and in some cases the history of mathematics as suggested by that poem. And then in turn mathematics can give deeper insight into the poem itself.

The mathematics as addressed in some of the poems is obvious (some of the titles are numbers like *-2, Zero,* and π). Some poems have mathematical equations as subtitles which are suggestive. Others end with mathematical equations. Some deal with the fascination of mathematical objects. Some deal with famous theorems or results. Others poked fun at mathematicians. Some poems are a starting point for further mathematical development.

There are a series of poems dealing with prime numbers. Another series considers numbers starting with the integers, through the irrational numbers, to the transcendental numbers and the computable numbers.

So maybe while there is no *Royal Road* to geometry or mathematics, poetry may provide a less bumpy route.

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E.R. Lutken grew up in the South with a family who loved music, poetry, and the outdoors (some of them also loved mathematics). She studied at Duke University and the University of Southwest Texas Medical School and completed her residency in family medicine. As a physician, Dr. Lutken worked first in urban emergency rooms and for a brief stint oversees, then for many years on the Navajo Nation. After that, she taught middle and high school science and mathematics and rural Colorado for several years. Now she spends her time reading, writing , messing around with math, playing music, and fishing in the swamps of Louisiana and the mountain streams of New Mexico.

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From the early Greek thinkers to today, mathematics ("clothes") was seen as the means to understand the world ("shaggy universe") around us though the way the Greeks used mathematics to understand the world is very different from today's usage. But in the end, as we extend our understanding of mathematics, it itself may be seen its own "savage deity".

This was the leading poem in *Manifold – poetry of mathematics*. The corresponding essay serves the same purpose for *Mathematical Riffs* by serving as an introduction to the other essays that follow.

In some sense this is a continuation of *Fundamentals of Mathematics s*tarting with the early Greek idea that *All is Number*. The final line is a list of five modern mathematical equations which are developed in the attached essay. The lines of the poem laid out in the form of a right triangle, a formatting feature that occurs in three other poems.

Zero: A poem about nothing. How can something which is nothing turn out to be so important?

One –The starting point for the integers is the ending stanza of the song "*Green Grow the Rushes O!"* Using the song, the integers 1 through 12 are presented (with a few important irrationals inserted in-between).

The initial most natural application of the integers is to count: *1, 2, 3, … infinity*? Cantor and his results about infinity (mentioned in *Fundamentals of Mathematics*) will play an important role in the next four essays.

Note: A *ghazal* is a poetic form of Arabic origin consisting of at least 5 rhyming couplets with each couplet having the same rhyming pattern (and meter): for example AA BA CA DA EA …

The problem of irrational numbers, their existence kept hidden by the Pythagorean brotherhood, their existence revealed by Hippasus and his punishment.

A continuation from *Irrational Loss*,- an introduction to some of the stranger irrational numbers beyond the constructable numbers like algebraic and transcendental numbers. Recalling *Cantor's Ghazal* and looking ahead to *Metempsychosis.*

2 is a rational integer, $\sqrt{2}$ is irrational and algebraic; $2^{\sqrt{2}}$ is transcendental.

Can machines think? Can computers be human? Alan Turing with the Turning Test proposed a method to answer this question. Continuing on, a Turing Machine (equivalent to a modern computer although much slower) defines *what is computable*. This in turn introduces a new class of numbers: computable numbers which as it turns out are only countably infinite (recall *Cantor's Ghazal*) meaning most numbers are not computable and therefore, perhaps (?), beyond human knowing?

How the words of a poem are arranged on a page can enhance the message of a poem (see *Triangular Numbers*). So it is with *Measured Illusion* whose stairstep form when rotated reflects the Euler - Mascheroni Constant, the theme of the poem and the theme of the essay that follows. This is the first of four famous numbers with ϕ, e, and π to follow!

Φ – Phi – The Golden Ratio – a most elusive number – bound to the Fibonacci sequence and spirals (natural and man-made) and even art (?).

e the base of the natural exponential function $y = e^x$ which models (?) Phaethon's Ride: flying too high: exponential growth; flying too low: exponential decay.

π is the ratio of a circle's circumference with its diameter: an overview of the history of this most famous constant.

A poem about a couple and loss: hence – 2. A famous story (myth) of another ancient couple. An introduction to binary (base 2) numbers.

Is 12 the best number? It's highly divisible and easy to work with, but what about a different base? We've considered base 2 – binary. How about octal (base 8) or hexadecimal (base 16) which in some sense are *derived* from binary (it's easy to convert between binary and octal/hexadecimal). Roman numerals are *biquinary* since they have separate symbols for fives (V, L, D) and ones (I, X, C, M). There is a mathematical *proof* for the most efficient base (which is close to 3), and that leads us to introduce balanced ternary (base 3) notation. Still base 12 has a lot going for it.

The lure of prime numbers: what is known and unknown about them. How to find one. Why they are so surprisingly useful!

The Sieve of Eratosthenes is an ancient technique to detect primes. Here we interleave Lutken's poem with the Sieve of Eratosthenes to find all primes less or equal to 100.

The arrangement of this poem is a *Tetractys* (see *Fundamentals of Mathematics*). This introduces a presentation of triangular numbers and figurate numbers.

A poem that ends with the most beautiful equation in mathematics: Euler's Identity which links 5 famous constants: π, e, 0, 1, and i. Also, an introduction into the *geometry* of complex numbers where we see multiplication by the number $e^{i\theta}$ is a rotation in the complex plane of the unit vector by θ radians!

Ramanujan and the "Taxicab number" 1729

The fascination of the Mandelbrot Set.

Two famous mathematical conjectures, only one of which has been proven true, and as to the other – there is no known proof – maybe it's true but unprovable?

From the movie "*The Good, the Bad and the Ugly*" – computing the probable outcome of a three-way duel or *truel*. An introduction to Monte Carlo techniques.

According to Wikipedia "A **clerihew** a whimsical, four-line biographical poem of a type invented by Edmund Clerihew Bentley. The first line is the name of the poem's subject, usually a famous person, and the remainder puts the subject in an absurd light or reveals something unknown or spurious about the subject. The rhyme scheme is AABB, and the rhymes are often forced. The line length and meter are irregular. Bentley invented the clerihew in school and then popularized it in books."

Here are eight **clerihews** of eight famous mathematicians.

The Ancient of Days – Watercolor by William Blake – 1794.

Is God a mathematician?

Fundamentals of Mathematics

Clothes we put on a shaggy universe
to make it behave, slick and comb its hair,
button its collar and send it to church.

Stack of bones extracted at autopsy,
bleach-soaked, messy bits removed,
arranged on a pedestal in the classroom,
shellacked, perfectly articulated,
clattering, ghoulish display.

Tome of mind-scrambling spells,
fanciful shapes teased from thin air,
mazes, knots, rotating matrices,
meandering möbius strips,
rippling surfaces, Klein bottles.

Book, bone, garb, trivial
intersections with mortal perception.

It is its own rough beast,
roaming an unknowable territory, staring
with omnichromatic vision through focal
depths of countless glittering ommatidia,
rambling amid tangles of helical strands
and galactic filaments, wild variables
crawling over its skin. It feasts on mushy
stew of particle and wave, gravity, time,
breathes in the spin of every lepton, laughs
at notions of elegance, structure, wisdom,
plays with axioms like toys.
Secret, savage deity

- E R Lutken (3: A Taos Press © 2021)

The Universe, Science, and Mathematics

"The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning." – Eugene Wigner, *The Unreasonable Effectiveness of Mathematics*, *Communications in Pure and Applied Mathematics* **13** (1960)

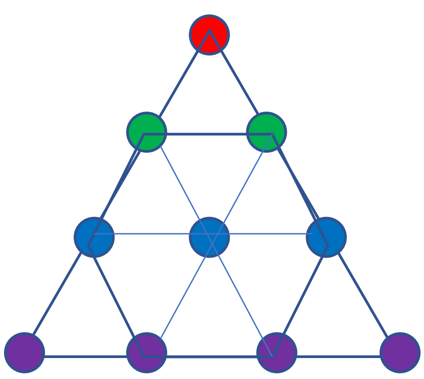
The enormous usefulness of mathematics in the natural sciences is something bordering on the mysterious. - Eugene Wigner

"Clothes we put on a shaggy universe
to make it behave, slick, and comb its hair,
button its collar and send it to church."

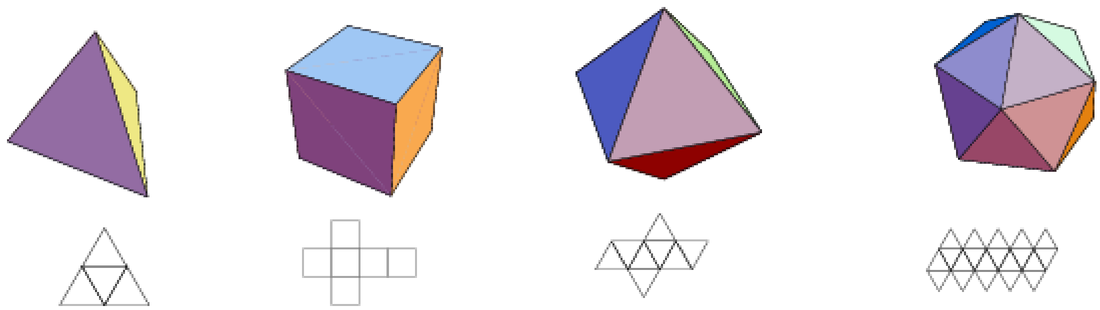
"All is Number" - Pythagoras

In western thought the idea that mathematics ("Clothes") was the key to understanding the "shaggy" universe has its documented origins with the Greek philosophers of the 5th and 6th centuries BCE. The Pythagoreans, the brotherhood founded by Pythagoras (ca. 570 – ca. 490 BCE.), believed that "all was number" or "the *substance* of all things was number". Their discovery that simple ratios like 2:1, 3:2, and 4:3 resulting in harmonic music intervals (an octave, a perfect fifth, a perfect fourth) reinforced this idea. But their approach to understanding the universe through number was a *qualitative* one and not *quantitative* one, the latter an approach to understanding that would not occur until the western Enlightenment in the 16th and 17th centuries CE. For the early Greeks numbers had *properties* which were *linked* to the natural world somewhat along the line that 666 was the "number of the beast".

Consider the Pythagorean *Tetractys* (left), a triangular arrangement of 10 points arranged in four rows of 1, 2, 3 and 4 points which can represent spatially a point (0 dimension), a line (formed by 2 points – 1 dimension), a plane (triangle formed by 3 points – 2 dimensions) and a solid (a tetrahedron, a solid figure formed by 4 points in 3 dimensions). Adjacent row ratios 2:1, 3:2, and 4:3 are the ratios for the musical octave, the perfect fifth, and the perfect fourth. The tetractys could also be linked with the four elements: fire, earth, air, and water.



In Plato's dialog *Timaeus* (ca. 350 BCE), four of the five so-called Platonic regular solids, tetrahedron, cube, octahedron, and icosahedron are identified with the four elements fire, earth, air, and water respectively based on their *geometric properties*. (Note, the existence of only five regular solids was proved in Euclid's *Elements* ca. 300 BCE as Prop. XIII.18.)

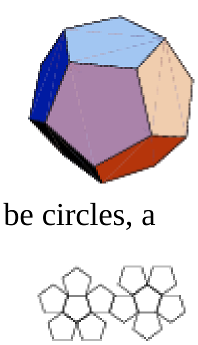


Tetrahedron (fire) | Cube (earth) | Octahedron (air) | Icosahedron (water)

The Tetrahedron (4 sided) was identified with fire because it was "sharp" like fire. The Cube (6 sided) was identified with earth because it was stable. The Icosahedron (22 sided) was water as it was the most "round" and therefore *fluid* like water. The Octahedron (8 sided) was air being somewhat between fire and water. Note that like the tetrahedron and the icosahedron, the octahedron had triangular sides. The shape of the mathematical object was identified with a property of that element.

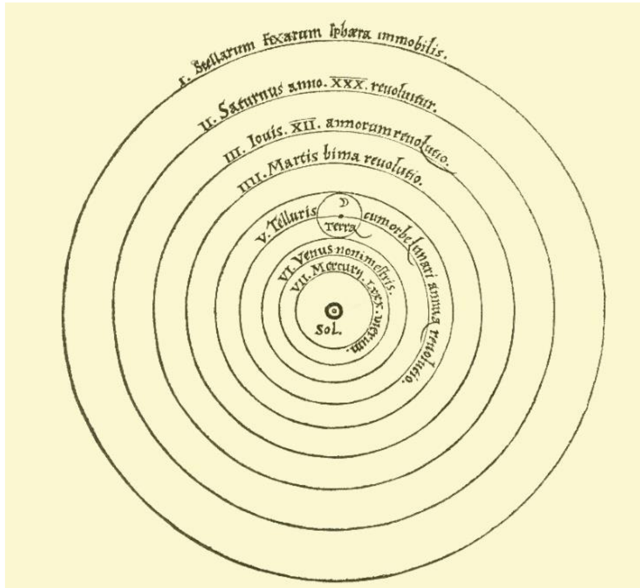
The 5th regular solid, the dodecahedron (12 sides), was assigned to the "universe" as a whole (which seems to me to be a bit of a "*kluge*"). It was identified with a 5th non-terrestrial element referred to as *aether* which made up the nonchanging heavens.

Since the heavens were seen as perfect, the orbits of the planets were considered to be circles, a perfect shape. Motion was *uniform*, not varying - again, a *qualitative* application of mathematics used to describe nature consistent with the Aristotelian notion of describing the natural world in terms of *qualities* or *causes*, not *quantities*.



Dodecahedron (aether)

From the Qualitative to the Quantitative



The standard *Ptolemaic* geocentric system placed the earth at the center of the universe with the sun, moon, and the five known planets revolving around the earth in circular orbits (circular because the circle was seen as perfect as the heavens were perfect.) Copernicus (1473-1543) proposed a simpler *heliocentric* model also using *circular orbits* and *uniform motion* of planets with the moon revolving around the earth and the earth and the five known planets revolving around the sun. Its advantage (?) was that it provided a cleaner and simpler model though interestingly enough, the Copernican system was no more accurate than the Ptolemaic. Needless to say, the *Copernican* system did not immediately replace the Ptolemaic system.

Today we understand that there were three fundamental flaws in understanding the motion of the planets: orbits were *circular*, motion was *uniform*, and the motion of a body required a *force physically acting* on it to move it (an outermost sphere enclosing the universe revolved causing the inner planets to move).

Galileo and Kepler

> "[The universe] cannot be read until we have learnt the language and become familiar with the characters in which it is written. It is written in mathematical language, and the letters are triangles, circles and other geometrical figures, without which means it is humanly impossible to comprehend a single word."
> *Opere Il Saggiatore* p. 171. - Galileo

Three 16th - 17th century figures, Galileo, Kepler, and Newton are important for understanding the shift to using mathematics to describe the universe *quantitatively*. Kepler's earlier then later works provide an interesting shift from a *qualitative* to *quantitative* mathematical description of planetary motion.

Johannes Kepler (1571 – 1630) in his work *Mysterium cosmographicum* proposed a model for the Copernican system based on the five Platonic solids which he used to explain the distances between the planet orbits. It has very much the favor of a *qualitative* mathematics.

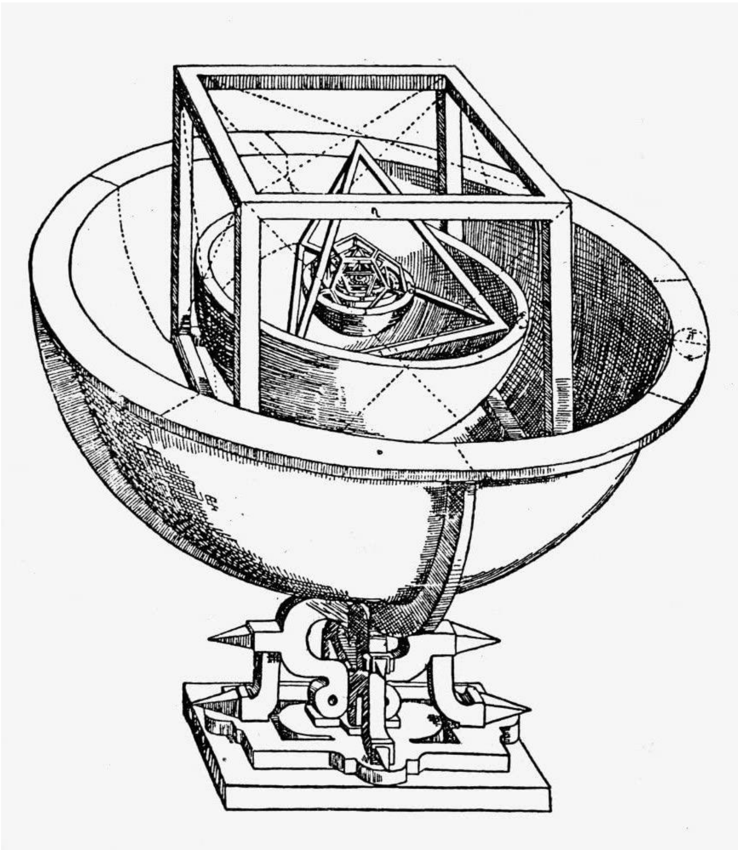
Starting with the outermost planet Saturn, in a sphere containing the orbit of Saturn, inscribe a cube (seen right). Within this cube inscribe a sphere: this is the orbit of Jupiter.

Within Jupiter's sphere, inscribe a tetrahedron (seen right). Within this tetrahedron inscribe a sphere: this is the orbit of Mars.

Within Mars's sphere inscribe a dodecahedron. Within this dodecahedron inscribe a sphere: this is the orbit of the Earth. Note that the Moon revolves around the Earth.

Within Earth's sphere inscribe an icosahedron. Within this icosahedron inscribe a sphere: this is the orbit of Venus.

Within Venus's sphere inscribe an octahedron. Within this octahedron inscribe a sphere: this is the orbit of Mercury.



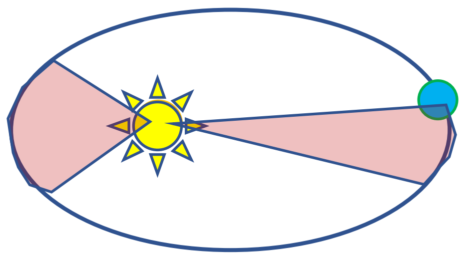
Platonic solid model from Mysterium cosmographicum

As Kepler saw it this must be correct since the earth and the five known planets are separated by the five Platonic solids – an example of the *qualitative* use of mathematics to explain the universe. Unfortunately, as it turned out the model did not quite fit the data as astronomical observations were getting better.

Better and more accurate astronomical observation (specifically the observed retrograde[1] motion of Mars as seem from the Earth) caused Kepler to revise (throw out?) his model. Here we see a shift from *qualitative* to a *quantitative* mathematical explanation of planetary motion.

In 1609 Kepler published his first two laws of planetary motion

> Law I (the Ellipse Law) - the curve or path of a planet is an ellipse (not a circle) whose radius vector is measured from the Sun which is fixed at one focus (a Copernican heliocentric system).



> Law II (the Area Law) - the time taken by a planet to reach a particular position is represented by the area swept out by the radius vector drawn from the fixed Sun (areas in pink). Note that this no longer assumes unform velocity.

Thus no longer circular orbits and no longer uniform motion! Kepler's 3rd Law came later.

> Law III (the Square Cube Law) For all planets the ratio of the squares of their period will be the same as the ratio of the cubes of the mean radii of their orbit.

Finally Newton

Isaac Newton (1642 Julian Calendar– 1727) proposed his law of gravity, a *mathematical equation*, which states all objects are mutually attracted with a force proportional to the product of their masses, $m_1 \cdot m_2$, and inversely proportional to the square of their distance, d^2, with G being a gravitation constant. That is $Force = G\dfrac{m_1 \cdot m_2}{d^2}$.This along with his three laws of motion …

> Law 1 – (Inertia) A body at rest stays rest and a body in motion stays in motion unless acted upon by an outside force (no need of an outside force to *maintain* movement) .
>
> Law 2 – Force is a function of mass times acceleration or $F = m \cdot a = m \cdot \dfrac{dv}{dt}$

[1] The retrograde motion of Mars was the apparent backwards motion of Mars as seen from Earth. With a smaller and faster orbit of Earth inside the larger and slower orbit of Mars, as the Earth overtook Mars, the position of Mars in Earth's sky was seen to go backwards.

Law 3 – For every action there is an equal and opposite reaction.

… solved the problem of describing planetary motion, a *quantitative* description describing how the planets (and by extension the entire natural world) worked. Mathematically Newton described *that* or more precisely mathematically *how* a certain force (e.g. gravity) works, not the *why*. As Newton put it "*Non fingo hypotheses*" – I do not feign hypothesis.

Stack of bones extracted at autopsy,
bleach-soaked, messy bits removed,
arranged on a pedestal in the classroom,
shellacked, perfectly articulated,
clattering, ghoulish display.

Tome of mind-scrambling spells,
fanciful shapes teased from thin air,
mazes, knots, rotating matrices,
meandering möbius strips,
rippling surfaces, Klein bottles.

Book, bone, garb, trivial
intersections with mortal perception.

Back to the Pythagoreans and forward to Cantor and Turing

Originally the Pythagoreans believed that all quantities (numbers) were *commensurate*; that is given any two numbers x and *y* there was a third number *z* that *measured* both; that is in more modern terms there are integers *m* and *n* such that $x = m \cdot z$ and $y = n \cdot z$. Thus $\frac{x}{y} = \frac{m \cdot z}{n \cdot z} = \frac{m}{n}$ or the ratio of any two numbers is a *rational* number. To continue, if y = 1 then it follows that $x = \frac{m}{n}$ so every number is a *rational* number, in the form $\frac{m}{n}$ for integers *m* and *n* where *n* does not equal zero.

Unfortunately, they also discovered that there were numbers which were not rational. For example, $\sqrt{2}$ cannot be expressed as the quotient of two integers (see entries for *Irrational Loss* and *Ode to* $2^{\sqrt{2}}$). These are the so-called *irrational* numbers, which together with the *rational* numbers make up the set of *real* numbers (see *Meditation on Transcendental Number*).

Then in the 19th and 20th centuries it was proved that…

1. The *rational* numbers could be put into a one-to-one correspondence with the natural numbers; that is, you could enumerate them or line them up one to one with the natural numbers 1, 2, 3, … etc. They were *countably infinite*. In other words, there are as many rational numbers as integers because you could *pair them off*.

However, Georg Cantor (1845-1918) proved that the set of all *real* numbers *could not* be put into a one-to-one correspondence with the natural numbers (or integers); they were *uncountably infinite* – a larger infinity. See *Cantor's Ghazal*.

For example, if you tried to pair off the integers (or even the *countably infinite rational numbers*) with all the real numbers, there would always be some real numbers left out. So, if the rational numbers are *countably infinite*, and the real numbers (rational and irrational) are *uncountably infinite*, the conclusion is that the *irrational numbers are uncountably infinite*. That is, in some *strong mathematical sense* there are more *irrational* numbers than *rational* numbers. Thus the *irrational* numbers make up the majority of all *real* numbers.

2. To continue, a *computable* number is one which can be computed out to any degree of accuracy using a Turing machine (Alan Turing 1912-1954) or its modern equivalent, a computer program (see entry for *Metempsychosis* which elaborates this idea). However, the set of Turing machines is *countably infinite*, hence the set of *computable* numbers (which easily includes all *rational* numbers and some (but not all) *irrational* numbers (like π) are therefore countably infinite which means …

3. Most *irrational* numbers cannot be computed (and are therefore *unreachable*? *unknowable*? beyond human *knowledge*?).

Thus Mathematics ….

"It is its own rough beast,
roaming an unknowable territory, staring
with omnichromatic vision through focal
depths of countless glittering ommatidia,
rambling amid tangles of helical strands
and galactic filaments, wild variables
crawling over its skin. It feasts on mushy
stew of particle and wave, gravity, time,
breathes in the spin of every lepton, laughs
at notions of elegance, structure, wisdom,
plays with axioms like toys.
Secret, savage deity"

"All is Number"

Birth of One
Lyre's clear song
Over foaming sea
+
Not two ships, 2
Peeled to the crux
Notes, 3rds, 4ths, 5ths
Reverberate in singing stars
=
Dung beetle marbles, Saturn's moons
Forest canopies, beehives, cracked clay
Cosmic nets, chromatophores, mayfly wings
Leptons, planets, whirlwinds, summersault spiders
$F = ma$, $i^2 = -1$, $V-E+F = 2$, $dS \geq 0$, $x_{n+1} = rx_n(1-x_n)$

- E R Lutken (3: A Taos Press © 2021)

"All is Number"

"All is Number" is attributed to the 6th century BCE Greek philosopher Pythagoras who proposed that somehow number was the key to understanding the world (recall *Fundamentals of Mathematics)*. The idea that mathematics is a means (or *the means*) to describe and understand the universe has come down to us today.

Originally mathematics was seen as a *qualitative* description of the world. For example, because circles were perfect and since the heavens were prefect, the orbits of the planets were circular. In a way this made sense especially since given the lack of better observational data, there was no evidence to the contrary. That would have to wait until the 17th century.

However, by the 17th century with the observational data provided by Tycho Brahe, Kepler suggested a different mathematical model, a *quantitative* description based on ellipses with orbits of planets determined by equal areas being swept out in equal times. This was simplified by Newton's Law of Gravity $F = G\frac{m_1 m_2}{r^2}$ which expressed the force of gravity F between two objects as a function of a constant G times the product of the two masses, m_1 and m_2 divided by the r^2, the square of the distance between them. It is important to note that this is a mathematical equation, a *quantitative* description describing the effect of gravity, a force, which in turn directed the orbits of the planets around the sun.

Thus, it is still true – "All is Number" but used now as a *quantitative* description of the world, not a *qualitative* one.

The poem's last line of *five very different equations* presents the richness of mathematics as seen in different areas of knowledge.

$F = ma$

Newton's Second Law of Motion: Force *F* equals Mass *m* times Acceleration *a* where $a = \frac{dv}{dt} = \frac{d^2s}{dt^2}$.

Acceleration is the *first derivative* (rate of change) of velocity *v* with respect to time *t* and the *second derivative* of distance *s* with respect to time *t (or the rate of change of the first derivative, velocity, with respect to time)*. Recall that Newton along with Leibniz was a co-discoverer (or co-inventor?) of calculus!

This equation models our understanding of the physics of motion.

$i^2 = -1$

Complex Numbers: If $i = \sqrt{-1}$ then $i^2 = \left(\sqrt{-1}\right)^2 = -1$

All positive numbers have two square roots. For example, $\sqrt{4} = +2, -2$. But what about the square roots of negative numbers, for example $\sqrt{-4} = ?$ or even $\sqrt{-1} = ?$ As mathematics and science advanced so our understanding of number also advanced, so to answer this square root question, we define $\sqrt{-1} = +i, -i$ so $\sqrt{-4} = \sqrt{4} \cdot \sqrt{-1} = 2i, -2i$ - two square roots!

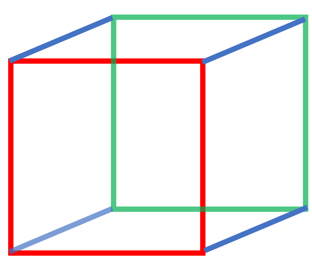
The addition of $i = \sqrt{-1}$ extends the set of numbers to include the so-called (and poorly named) *imaginary numbers*, for example $2.7182i$. So the set of *real numbers* like *3.1415* when combined with (added to) the set of *imaginary numbers* yielded the set of *complex numbers* like $3.1415 + 2.7182i$ thus opening a whole new understanding of numbers. See *Meditation on Transcendental Numbers*, *Euler's Identity*, and *Augury in Sand.*

V – E + F = 2

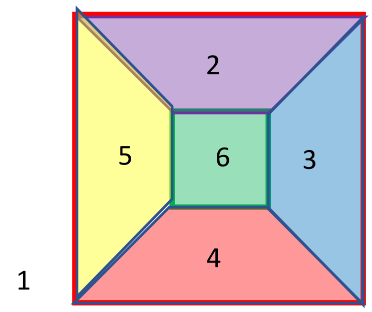
V – E + F = 2, Euler's Polyhedral Formula (see *Distillations*) is a ***topological*** formula about solid three-dimensional polyhedrons which has applications in networks and graph theory. *Topology* is the study of the properties of geometric figures which are invariant under continuous transformations meaning you can stretch, twist, and bend but not break the figure. This is sometimes referred to as *rubber-sheet geometry*. In this case given any polyhedron (with no holes in it), the number of vertices V minus the number of edges E plus the number of faces F always equals 2.

For example, a cube is a polyhedron having 8 Vertices, 12 Edges and 6 Faces.

Therefore 8 – 12 + 6 = 2.



A polyhedron can be flattened into a two-dimensional planar graph. For example, if you take the cube above, stretch it and flatten it out so that the area bounded by the red face above is stretched and pulled back to be the outside boundary of the red, yellow, purple, blue, and center green regions (on the right) then this planar Map of Oz also satisfies the V – E + F = 2 equation.



There are 8 vertices, 12 edges and 6 faces (or regions) so

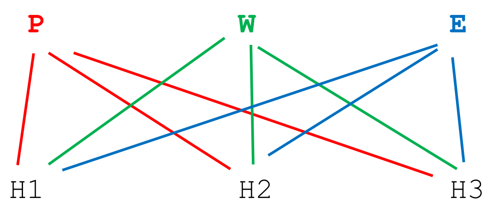
$$V - E + F = 2 \ or \ 8 - 12 + 6 = 2$$

Thus, the V – E + F = 2 result holds for all planar graphs counting the outside region as one face.

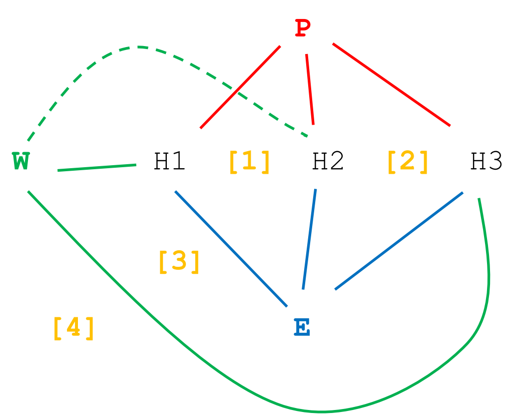
In fact, every planar graph has the Euler V – E + F = 2 property counting the exterior region as one of the faces. If a graph does not satisfy V – E + F = 2 then the graph cannot be planar.

What is interesting is that *any graph which does not satisfy V – E + F = 2 cannot be planar* has applications in theory of networks and graph theory.

For example, consider the Three Utilities Problem: Given three utilities, Phone, Water and Electricity and three Houses, is it possible to connect each of the three houses to each utility without having the lines crossing (a planar graph) ?



V = 6. E = 9 (each of 3 houses is connected to 3 utilities). However, F (or the number of regions) is a bit more difficult to figure out. Below can see that a loop with four edges (utility to house to different utility to different house and back) defines a region or face (e.g. P to H1, H1 to E, E to H2, H2 to P or P(H1)E(H2)). So how many four-loops are there?



Ignoring the W to H2 edge (dashed green) we can easily count four regions defined by solid lines: [1]: P(H1)E(H2), [2]: P(H2)E(H3), [3]:W(H1)E(H3), and the outside region [4]:P(H3)W(H1). But what about the W to H2 dotted-edge?

The way to count the 4-loop regions follows.

Each 4-loop has either {P, E}, {P, W} or {E,W} as two of its vertices with the other two obtained from the set {H1, H2, H3}.

The P-H loops: You can do P(H1)E(H2), P(H1)E(H3) or P(H2)E(H3) – for only three unique 4-loop regions! Careful because there are multiple orderings to describe the same 4-loop regions which is why this is complicated.

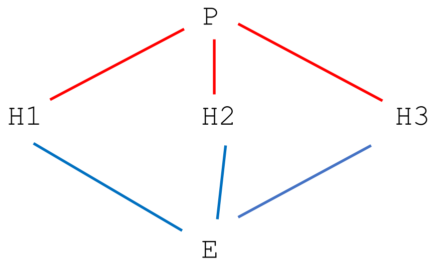
The P-W loops: In an equivalent way, you have P(H1)W(H2), P(H1)W(H3) or P(H2)W(H3) for three more unique 4-loop regions.

The E-W loops: And finally, you have E(H1)W(H2), E(H1),W(H3), or E(H2)W(H3) for three more unique 4-loop regions.

Altogether the number of faces (loops) is 9 but V – E + F = 6 – 9 + 9 = 6 not 2!

So, the take-away is the Three Utilities Problems has no planar graphical solution since the number of vertices, edges, and faces (regions) violates Euler's Polyhedron Formula.

Consider a simpler case with 2 utilities (Phone and Electricity) and 3 houses. With one pair of utilities and 3 pairs of houses, the number of loops (and therefore regions) of utility to house to different utility to different house is $1 \times 3 = 3$.



Here we see a planar graph with 5 Vertices, 6 Edges and 3 Faces or Regions: P(H1)E(H2), P(H2)E(H3) and P(H1)E(H3) so V – E + F = 5 – 6 + 3 = 2

Euler's Polyhedron Formula expresses a *topological feature* about planar graphs as a *numeric relationship between the number of edges, vertices and faces*.

$dS > 0$

This very simple inequality is the 2nd Law of Thermodynamics which states that the *change in entropy S* (denoted *dS*) is always positive. The universe is increasing in disorder; things run down over time. Note how a simple mathematical inequality describes a scientific observation.

$$x_{n+1} = r \cdot x_n \cdot (1 - x_n)$$

A *recurrence relation* like $x_{n+1} = r \cdot x_n (1 - x_n)$ is an example of a *feed-back function* where given a variable x_n (starting with some initial x_0) a new value x_{n+1} is computed which in turn is used to obtain the next value x_{n+2} and so on. It's useful if the sequence of x's converges to some value.

For example, the square root of any positive number *a* can be computed using the recurrence relation $x_{n+1} = \frac{x_n^2 + a}{2x_n}$ which given any starting value for x_0 *close to* $\sqrt{a}$ converges quickly to $\sqrt{a}$.
The recurrence relation given above is Newton's Iteration formula for computing the square root of *a* > 0.

Another way of thinking about a *recurrence relation* is that describes a quantity that can be expressed (or defined) in terms of a *smaller case of itself*. (Note there must be an initial value x_0 to prevent the problem of *infinite regression*!)

The logistic function $f(x) = r \cdot x(1 - x)$ is a family of recurrence relations depending on parameter *r* whose behavior depends on the value of *r*. It is used to demonstrate *chaos* or *chaotic behavior* (see *Math History in a Few Bad Clerihews*). If *r* is less than 3.0 then any initial value of x_0 such that $0 < x_0 < 1.0$ will eventually converge to a single value although the closer *r* is to 3 the slower the convergence. Things get interesting for values of $r > 3.0$.

Demonstrating Chaos : Web Diagrams

Chaotic behavior (see *Math History in a Few Bad Clerihews*) is best seen graphically with a web diagram. The diagrams on the following page start with plotting the logistic function $f(x) = r \cdot x(1 - x)$ (in red) on the interval of interest, in this case the closed interval $[0,1]$. Also displayed is the line $y = x$ (also in red).

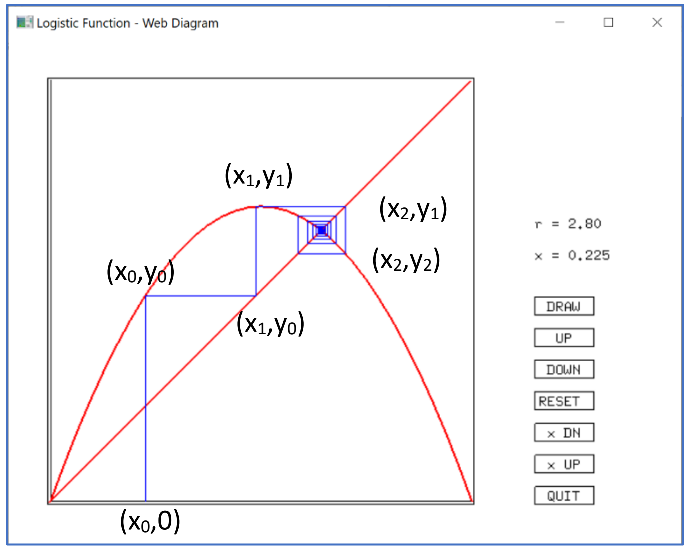
An initial value x_0 is chosen at random. The corresponding $y_0 = r \cdot x_0 (1 - x_0)$ is computed and a *vertical line* segment (in blue) from $(x_0, 0)$ to (x_0, y_0) is drawn. Since $y_n = r \cdot x_n (1 - x_n)$ is a quadratic function, the values (x_n, y_n) will plot on the upside-down parabola of the logistic equation. Since we are plotting the action of a recursive function we set x_1 equal to y_0, draw a *horizontal line* from (x_0, y_0) to the line $y = x$ at (x_1, y_0), and next recursively compute $y_1 = r \cdot x_1 (1 - x_1)$ since the x-coordinate on this line equals x_1. As before we draw the *vertical line* from (x_1, y_0) to (x_1, y_1) obtaining a second value for the recursive function . Repeating this process draws a web. *Each new y value becomes the next x value*.

For example, if *r = 2.8* then for any initial value of x_0 (where $0 < x_0 < 1.0$) the iteration of $x_{n+1} = r \cdot x_n (1 - x_n)$ will eventually converge to x = 0.6428571429

In the Web Diagram to the right we plot the ordered pairs (x_n, x_{n+1}), (x_{n+1}, x_{n+1}) etc. generated by the recurrence relation.

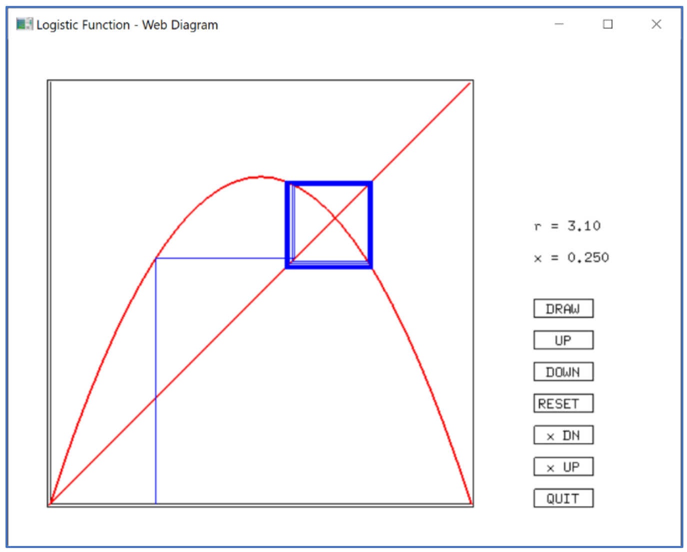
Note how they spiral in to the fixed value 0.642857.

Note how the values converge to a single point on the parabola – intersecting the line y = x.



At $r = 3$ bifurcation occurs – the values of x_n converge to two values bouncing back and forth between two points on the parabola. And as *r* increases further bifurcations occur to values with period 4 then 8 etc. to eventually *chaotic* behavior.

It should be pointed out that no matter where we start, we always converge to the same point(s) on the parabola.

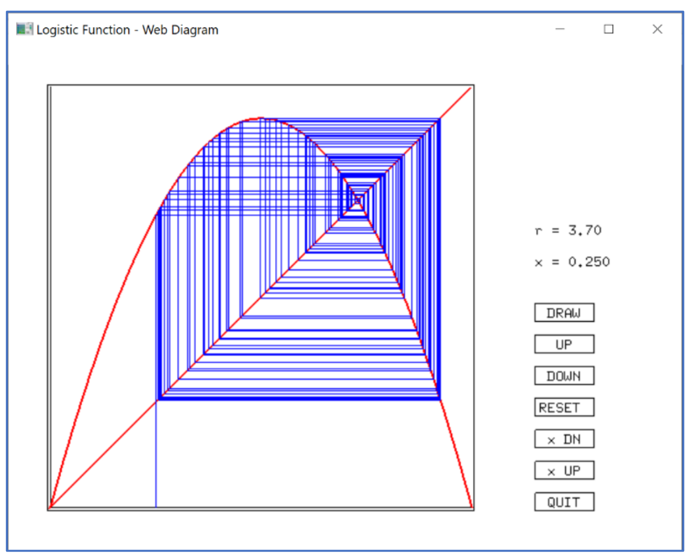


At r = 3.70 we see the values of x_n chaotically bounce over a range of values.

It is chaotic because if we alter the initial value of x, (in this case 0.250), the resulting web will be very different.

This is called *sensitivity to initial conditions*, a feature of chaotic behavior which makes the long-term behavior of chaotic phenomena (like the weather) difficult to predict.

Note that as *r increases* the slope where the parabola intersects the line y = x becomes *steeper* – a hint (?) as to why chaos occurs?



Zero

Zip, null, the empty sum.

What is the use of a number
that cannot be?
Not prime, not composite,
not positive or negative.

No one can sense it –
it's nothing.

Can't divide it up –
all parts of nothing are nothing.

Can't use it to divide things either –
that would mean cutting something into no pieces.

Multiplying it doesn't do jack –
if it isn't anything to start with
why would more of it be something?

So what's the use?
Place holder? Identity marker?
Symbol of absence? Point of departure?

0

opens on the page like the dark mouth of infinity,
hobbit door to a blank symmetry
where fragile constructs of points, vectors, strings dance,
arcs, lines, planes stretch to no end
and all ends. - E R Lutken (3: A Taos Press © 2021)

Zero: A Sign Found in the Boston Museum of Science

Zero is the Cardinal Number of the Empty Set

It's the number of things you have when you don't have anything

It's the most effective computing device ever invented

Without it we could not distinguish 23 from 20030 or 20300 or 23000 and arithmetic would be a good deal tougher than it is

And now look at this

Addition	$2 + 0 = 2$	$3 + 0 = 3$	$4 + 0 = 4$	and so on
Multiplication	$2 \times 1 = 2$	$3 \times 1 = 3$	$4 \times 1 = 4$	and so on

0 does the same thing for addition that

1 does for multiplication

The Problem of Zero

Is zero a number? Historically zero has presented problems:

> "What is the use of a number
> that cannot be?"

How can you count something that is not there – and if so, can it share the properties of or *behave like* the other numbers?

> "Not prime, not composite,
> not positive or negative."

The ancient Greeks did not consider zero to be a number. And strictly speaking, 1 was not a number either as a number was defined as a *multiplicity of unity* (i.e.1). Nevertheless, four different cultures (none were Western/European) developed a zero: Babylonian, Chinese, Indian (Southern Asia), and Mayan – mostly as a *placeholder* notational device.

> "So what's the use?
> Place holder? Identity marker?
> Symbol of absence? Point of departure?"

Yes, zero as *placeholder* or as the Sign says:

Without it we could not distinguish 23 from 20030 or 20300 or 23000 and arithmetic would be a good deal tougher than it is

The *placeholder use* for zero requires a certain type of number representation as in our modern positional notation where the position of a digit indicates its weight as a power of 10.

A number representation which had no need for zero is found in Roman Numerals. XXIII (23) is distinguishable from MMXXX (2300) which is distinguishable from MMXXX (2030) which is distinguishable from MMIII (2003). And 20030 could be expressed as $\overline{XX}\,XXX$ where the bar was used to multiply by 1,000. Zero as placeholder is not needed. However, Roman Numerals are awkward to use for calculation.

But does zero as placeholder, as a purely notational device, make zero a number? What would it take to make zero a real number? How can Pinocchio become a real boy?

Enter Zero as a Number

In the 7th Century C.E. the Indian mathematician Brahmagupta (see *Math History in a Few Bad Clerihews*) considered *zero as a number* and expressed how it worked with (or *played well with*) other numbers. He defined *zero as subtracting a number from itself*; that is if n is a number, then $0 = n - n$. He went on to state that …

> *When zero is added to a number or subtracted from a number, the number remains unchanged; and a number multiplied by zero becomes zero.*

An Aside: Today we say that 0 is the *additive identity*; that is, given any number n, there is a unique number 0 such that $n + 0 = n$ and $0 + n = n$. Furthermore, given any number n there is a unique number denoted ($-n$), the *additive inverse*, such that $n + (-n) = 0$.

Of course, we just write this as $n - n = 0$

The rule for multiplication by zero, that is $n \times 0 = 0$, *is* significant as we will see below.

More rules for zero were given in terms of *fortunes* (positive numbers) and *debts* (negative numbers). Note that multiplication by zero results in zero.

> *A debt minus zero is a debt.*
> *A fortune minus zero is a fortune.*
> *Zero minus zero is a zero.*
> *A debt subtracted from zero is a fortune.*
> *A fortune subtracted from zero is a debt.*
> *The product of zero multiplied by a debt or fortune is zero.*
> *The product of zero multiplied by zero is zero.*

He went on to state rules for multiplication and the division of positive and negative numbers, again in terms of fortunes and debts.

The product or quotient of two fortunes is one fortune.
The product or quotient of two debts is one fortune.
The product or quotient of a debt and a fortune is a debt.
The product or quotient of a fortune and a debt is a debt.

Finally, Brahmagupta *tried* to define division *by* zero.

Positive or negative numbers when divided by zero is a fraction with the zero as denominator.

Zero divided by negative or positive numbers is either zero or is expressed as a fraction with zero as numerator and the finite quantity as denominator.

Zero divided by zero is zero.

Why You Can't divide By Zero

"Can't divide it up –
$2 \times 0 = 0$ all parts of nothing are nothing.

Can't use it to divide things either –
that would mean cutting something into no pieces.

Multiplying it doesn't do jack –
if it isn't anything to start with
why would more of it be something?"

As noted above, multiplication by zero yields zero: For example: $6 \times 0 = 0$

Now if (since) Division is the *inverse* of Multiplication, for example $6 \div 3 = 2$, because $3 \times 2 = 6$, then $6 \div 0 = n$ if and only if $n \times 0 = 6$ except $n \times 0 = 0$ (not 6!). So, division *by* 0 is undefined.

Other Appearances of Zero in Expressions – Maybe some not so obvious

Aside from the fact that you cannot divide by zero (although zero *divided* by any non-zero number is 0; that is $\frac{0}{n} = 0$ for $n \neq 0$), zero plays well with other numbers; it *fits the pattern of how numbers work*. Consequently, we can extend the appearance of zero into other mathematical expressions.

Exponentiation: $b^0 = 1 \ \textit{for} \ b \neq 0$

For a positive integer *n* and $b \neq 0$, $b^n = \overset{<------n------>}{b \times b \times b \times ... \times b}$; that is b multiplied by itself n times. In the same way for a positive integer m $b^{-m} = \frac{1}{b^m}$. Using the standard (cancellation) rules of exponents,

$b^n \cdot b^{-m} = \frac{b^n}{b^m} = b^{n-m}$. If $n = m$ the numerator and denominator being equal cancel to 1 thus $b^{n-m} = b^0 = 1$. It fits the *pattern*.

Factorials: $0! = 1$? For any positive integer *n*, *n factorial* written $n! = n \cdot (n-1) \cdot (n-2) \cdot ... \cdot 3 \cdot 2 \cdot 1$ is the *product* of the integers 1 up to *n*. For example, $3! = 3 \cdot 2 \cdot 1 = 6$ and $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ etc.

However note that $2! = 2\times1!$, $3! = 3\times2!$, $4! = 4\times3!$ and in general $n! = n\times(n-1)!$ for any positive integer n *greater than* 1 (a *recursive formula* for factorials). But what about $1! = 1\times0!$? *Zero factorial*? As it turns out it's convenient to *define 0! equal to 1*. In a sense it *fits the pattern*. So, the recursive formula $n! = n\times(n-1)!$ holds for all positive integers *including* 1.

The **Binomial Coefficient** $\binom{n}{k}$ for non-negative integers k and n where $0 \le k \le n$ is defined as the number of different subsets of k object that can be selected from a set of n objects.

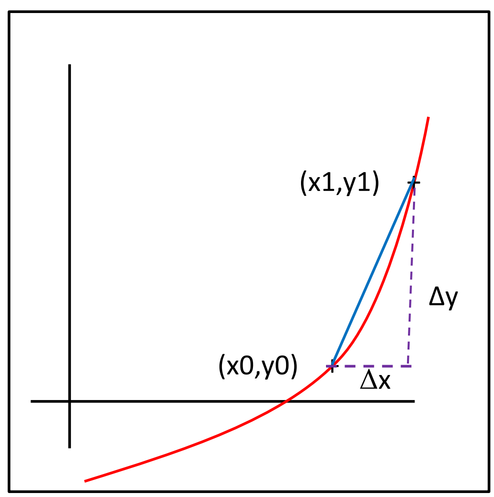
For example, $\binom{4}{3} = 4$ is the number of subsets of three items chosen from a set of four, $\{a,b,c,d\}$; that is

$$\{\{a,b,c\},\{a,b,d\},\{a,c,d\},\{b.c.d\}\}.$$

A formula for $\binom{n}{k}$ is given by $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ so $\binom{4}{3} = \frac{4!}{3!\times1!} = 4$. And since by definition $0! = 1$ it follows that $\binom{n}{0} = \frac{n!}{0!(n-0)!} = 1$ which makes sense since there is *only one empty set* that can be chosen from a non-empty set of n items. It *fits the pattern*.

The $\frac{0}{0}$ form and differentiation (for those familiar with calculus)

We've shown that you cannot *divide zero by zero* (Brahmagupta had trouble with this) but as a ***form*** and *not* a legitimate mathematical expression, $\frac{0}{0}$ is important and useful as seen in how it appears in calculus to define a derivative.



The Derivative:

The derivative from calculus is the *slope of a curve at a point*.

The slope of a straight non-vertical line is found by taking the coordinates of any two points (x_0, y_0) and (x_1, y_1) on the line, and taking the ratio of the change of y, Δy, divided by the change of x, Δx, to compute the slope $m = \frac{\Delta y}{\Delta x} = \frac{y_1 - y_0}{x_1 - x_0}$. Since the *slope of a straight line is the same everywhere,* it doesn't matter which two points you choose; the slope is always the same. From here is it not difficult to derive the slope-intercept equation of a line, $y = m\cdot x + b$, where the line crosses the y-axis at the *y-intercept* (0,b),.

But what about choosing two points on a curve (seen in the diagram above) and asking the question what is the slope of the curve at the point $(x_0.y_0)$?

To find the slope of the curve *at the point* $(x_0 y_0)$, using a second point (x_1, y_1) we take the ratio $\frac{y_1 - y_0}{x_1 - x_0}$ and sneak up on it by sliding the point (x_1, y_1) closer and close to the point (x_0, y_0). The slope is the limiting value as the two points get closer and closer. (Note – a typical calculus course covers limits first.)

To introduce some algebraic notation to this process, let the curve be described by some function $y = f(x)$ and now rephrase the question as *what is the limit as* x_1 *approaches* x_0 or

$$\lim_{x_1 \to x_0} \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

Notice that *in the limit* we're approaching the $\frac{0}{0}$ form.

Now if we let $h = x_1 - x_0$ and $x_1 = x_0 + h$ we obtain the standard definition for the *derivative of a function f(x)*:

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \quad \text{provided the limit exists.}$$

As *h* approaches zero, the so-called difference quotient $\frac{f(x+h) - f(x)}{h}$ approaches $\frac{0}{0}$. Yet as we approach an undefined ratio $\frac{0}{0}$, division by 0 as the *limiting value* of this ratio, the value of the ratio *close to h = 0,* may approach a definite value and if it does, we define this limiting value of be the *slope of the curve*. This is the trick! The target $\frac{0}{0}$ is undefined but as we get closer and closer to the target, the limit may be defined! As my high school math teacher would say

"we're sneaking up on division by zero!"

Example: Let $f(x) = x^2$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h \to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} = \lim_{h \to 0} \frac{\cancel{h}(2x+h)}{\cancel{h}} = \lim_{h \to 0} 2x + h = 2x$$

That is $\frac{d}{dx}(x^2) = 2x$

Thus, the $\frac{0}{0}$ ***form*** appearing in a limit is the basis for differentiation. Every function for which this limit exists is a differentiable function, a very important and large class of functions used to solve many problems.

To continue, once you have a derivative function defined, $f'(x)$, many problems using derivatives require finding where $f'(x)$ *equals 0*.

Example: Given the quadratic function $f(x) = ax^2 + bx + c$ (a parabola for $a \neq 0$) its derivative is $f'(x) = 2ax + b$. Setting the derivative equal to zero and solving $0 = 2ax + b$ yields $x = \frac{-b}{2a}$. This is the x coordinate for the *vertex* of the parabola (the y coordinate is $\frac{-b^2 + 4ac}{4a}$), the valley point (minimum) of the quadratic if a > 0 (the parabola opens up) or the peak point (maximum) of the quadratic if a < 0 (the parabola opens down).

Aside: While differentiation and calculus were developed in the late 17th century, it wasn't until the early 19th century that mathematicians put differentiation and calculus on a mathematically firm foundation. They knew it worked; they had a hard time *rigorously* justifying why it worked! In mathematics it's all about proof !

Searching for Zeros

If you take a course in algebra, you learn to use the quadratic formula $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ to solve the quadratic equation $ax^2 + bx + c = 0$; where the curve crosses the x-axis.

You also learn techniques for finding the zeros for any polynomial equation; that is, for what value of x is the polynomial expression $a_n x^n + a_{n-1} x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ equal to 0?

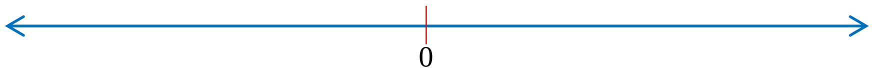
If you take a course in calculus, a standard problem is to find where the derivative of a function (see above) is zero. Why? Because that's where the function achieves a maximum or minimum value. That is where the function representing some mathematical model reaches its optimal value (where the rate of change $f'(x) = 0$). A lot of mathematics seems to come down to a search for zero.

> "What is the use of a number
> that cannot be?"

Good question since in algebra or calculus we're always looking for zeros!

So what is zero and what use is it?

"So what's the use?
Place holder? Identity marker?
Symbol of absence? Point of departure?



Zero is the center and origin of the real number line, the Cartesian plane and 3-space with x, y, z axes radiating in three spatial dimensions. The goal of a count-down: 10, 9, 8, 7, 6, 5, 4, 3, 2, 1, 0!

opens on the page like the dark mouth of infinity,
hobbit door to a blank symmetry
where fragile constructs of points, vectors, strings dance,
arcs, lines, planes stretch to no end
and all ends."

Is zero the "the dark mouth of infinity" or infinity's nemesis?
Zero: the center and starting point of all things.
What about the $0\times\infty$ *form? Does 0 destroy* ∞ *or does* ∞ *destroy 0?*
The Answer?
It depends!

"Who Knows One?"

Babylon's dawn, mark on a rough clay tablet,
born whole, natural, rational, real.
Single stroke of pen on primal parchment,
scratch of beginning, omneity unveiled
side by side with nothing, strings of words
map all matter ever known or thought.
Patterns portray a splendid, thorny
world revealed in sloppy truths and graceful knots.
The number that bears all, generously lends
itself, the source, the hum, the living wealth
of everything and never alters or bends
whatever is, but lets it be exactly itself.
One is One and all alone
and evermore shall be so

-- E R Lutken (3: A Taos Press © 2021)

Green Grow the Rushes, O

The last two lines come from the poem/song "*Green Grow the Rushes, O"* whose verses count backwards from twelve to one linking each integer with a collection of objects.

I'll sing you twelve, O
Green grow the rushes, O
What are your twelve, O?
Twelve for the twelve Apostles
Eleven for the eleven who went to heaven,
Ten for the ten commandments,
Nine for the nine bright shiners,
Eight for the April Rainers.
Seven for the seven stars in the sky,
Six for the six proud walkers,
Five for the symbols at your door,
Four for the Gospel makers,
Three, three, the rivals (arivals?),
Two, two, the lily-white boys,
Clothed all in green, O
One is one and all alone
And evermore shall be so.

The references in the poem/song seem to be a mixture of objects or groupings, both religious and astronomical. In any case as near as can be determined, the exact meaning of many of the verses can't be pinned down; some are obvious, others obscure.

Normally we think of integers as counting items. In the above poem/song each integer is closely identified with a particular group, object or notion. But as in the poem/song above, in the listing of the integers one through twelve below, many integers also have other algebraic, geometric, and cultural properties.

The integers 1 thru 12 with three interesting irrational numbers inserted in-between.

One – The Source of all Numbers

One is one and all alone
And evermore shall be so.

"*Single stroke of pen on primal parchment*": $| \rightarrow$ I $\rightarrow 1$

Since numbers referred to a *multiplicity*, the ancient Greeks did not consider one (*unity*) to be a number since numbers were seen as *multiplicities of unity*.

One is the multiplicative identity since $n \times 1 = 1 \times n = n$ for any number n.

A prime integer is an integer whose only divisors are 1 and itself. However, though the only divisors of 1 are 1 and itself, 1 is not considered prime. An important (and useful) theorem in mathematics is the Prime Factorization Theorem (see *Prime Syllabus Song*) which states that every positive integer can be *uniquely* expressed as a product of primes. If 1 were a prime, *uniqueness* would disappear.

On the other hand, 1 is considered to be a *factor* for any positive integer since 1 divides evenly into that integer. For example, the factors of 6 are 1, 2, 3, and 6 itself. It's just that 1 is not considered to be a *prime factor*.

1 is its own square: $1^2 = 1$, its own cube $1^3 = 1$ etc.

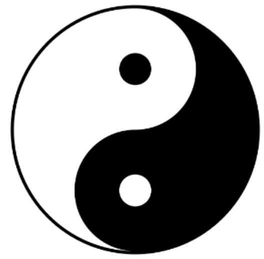
One is One and all alone
and evermore shall be so

Phi - $\phi = \dfrac{1+\sqrt{5}}{2} \approx 1.618033989...$

See *Φ*

Two

Two, two, the lily-white boys,
Clothed all in green, O



The smallest and only even prime.

Two points determine a line and the intersection of two lines determine a point.

Two is the dimension of a plane.

Two is the cardinality of Yin and Yang: the concept of duality (opposite or complementary forces).

Euler's Formula V- E + F = 2 See *All Is Number*

The base of binary notation: There are 10 kinds of people who understand binary (an old joke!).

Even integers are defined by being evenly divisible by 2; that is an integer n is even if and only if there is another integer k such that $n = 2 \cdot k$

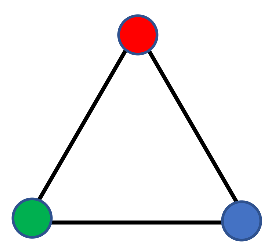
The Exponential e: $= \sum_{n=0}^{\infty} \frac{1}{n!} \approx 2.71828128...$

See *Phaeton's Ride*

Three

Three, three, the rivals (arrivals?),

The Three Wise Men (the arrivals?)



The first odd prime,

The 1st non-trivial triangular number ($1+2=3$).

Number of sides and angles in a triangle.

The dimension of space: length , width, height

An equilateral triangle can be constructed using only straight-edge and compass: From Euclid *Elements* (ca 300 BCE) Prop I.1 "To construct an equilateral triangle on a given finite straight line."

The Trinity

RGB (Red-Green-Blue) color triad used by computers.

$$\textbf{Pi - } \pi = \frac{circumference}{diameter} \approx 3.141592654...$$

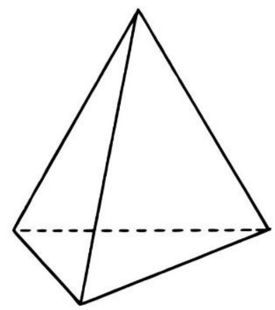
See π

Four

Four for the Gospel makers

The Four Gospels: Matthew, Mark, Luke , John

The first non-trivial square: $2^2 = 4$.



The number of vertices and faces in a regular tetrahedron (each face is an equilateral triangle).

A square can be constructed using only straight-edge and compass: From Euclid *Elements* (ca. 300 BCE) Prop I.46 "To describe a square on a given straight line"

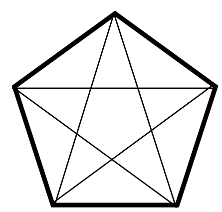
The four points of a compass: North, East, South, West

Four quadrants of the cartesian plane

Five

Five for the symbols at your door,

Five for the symbols at your door is obscure. (a pentagram?)



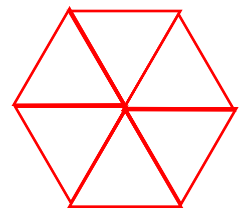
The 3rd prime number.

A regular pentagon can be constructed using only straight-edge and compass: From Euclid's *Elements* (ca. 300 BCE) Prop IV.11 "To inscribe an equilateral and equiangular pentagon in a given circle."

There are 5 platonic solids. Polyhedra all of whose faces are regular polygons: tetrahedron, octahedron, cube, dodecahedron, icosahedron. The dodecahedron has 12 pentagon sides, the cube square sides and the other three triangular sides.

Six

Six for the six proud walkers,



The first perfect number whose *proper* divisors 1, 2 and 3 sum to 6.

The 2nd non-trivial triangular number: $1+2+3=6$

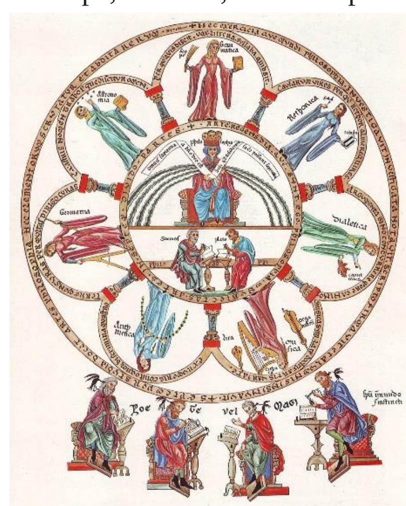
A regular hexagon can be constructed using only straight-edge and compass: From Euclid's *Elements* (ca. 300 BCE) Prop IV.15 "To inscribe an equilateral and equiangular hexagon in a given circle."

A cube has 6 square sides.

Seven

Seven for the seven stars in the sky,

The Pleiades aka the Seven Sisters: Maia, Electra, Alcyone, Taygete, Asterope, Celaeno, and Merope.



Philosophia et septem artes liberales, "philosophy and the seven liberal arts." From the Hortus deliciarum of Herrad of Landsberg (12th century)

7 is the 4th prime. 3, 5, and 7 make up a prime triplet.

The number of days in a week.

The number of observable celestial objects: Sun, Moon, Mercury, Venus, Mars, Jupiter, Saturn (hence the reason behind the 7-day week and the names of the days).

The seven days of creation.

The seven *classical* liberal arts (see right): the quadrivium (arithmetic, geometry, music, astronomy) and the trivium (grammar, logic, rhetoric)

Unlike a triangle, square, regular pentagon, and regular hexagon, a regular 7-sided figure (heptagon) cannot be constructed using only straight-edge and compass.

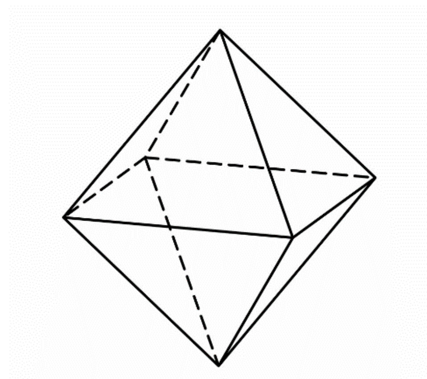
7 is a lucky number!

Eight

Eight for the April Rainers.

The Hyades star-cluster which rising in April signals rain

The first non-trivial cube; $2^3 = 8$.



The number of faces in a regular octahedron (with 6 vertices).

The number of vertices in a cube (with 6 faces).

The 8th day of creation (the New Creation).

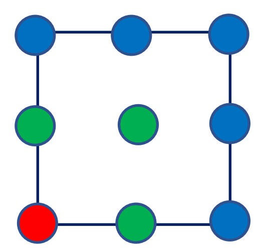
The 8-fold path of Buddhism

Nine

Nine for the nine bright shiners,

The Nine orders of Angels

The 2nd non-trivial square: $3^2 = 9$



$3^2 + 4^2 = 5^2$ or $9 + 15 = 25$ - the smallest Pythagorean triple

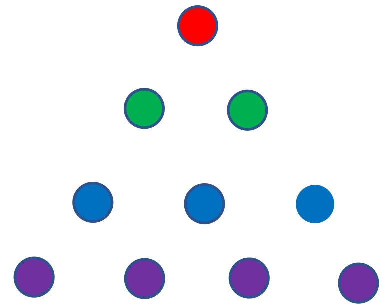
Repeatedly summing the digits of any decimal integer will eventually result in an integer value mod 9 which is its *remainder* when dividing by 9.
Example 2357 = 2 + 3 + 5 + 7 = 17 mod 9 = 1 + 7 = 8 mod 9
If the sum of the digits sums to 9, that number is a multiple of 9. *Casting out Nines* is a technique that can be used to *check* any arithmetic calculation.

For example: 214 + 467 = 681
214 mod 9 = 2 + 1 + 4 = **7**
467 mod 9 = 4 + 6 + 7 = 17 and 17 mod 9 = 1 + 7 = **8**
Adding **7** + 8 = 15 and 15 mod 9 = 1 + 5 = **6**.
681 mod 9 = 6 + 8 + 1 = **6**. Check!

The 2nd non-trivial square: $3^2 = 9$

Ten

Ten for the ten commandments,



The 3rd non-trivial triangular number: $1 + 2 + 3 + 4 = 10$

The base of our decimal numbering system.

The Pythagorean Tetractys – 10 dots arranged in a triangle (right) – see *Fundamentals of Mathematics.*

The 10 Commandments

Eleven

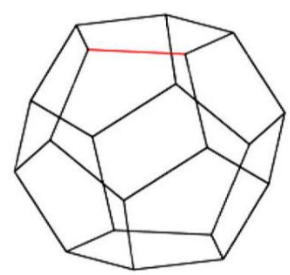
Eleven for the eleven who went to heaven,

The 12 original disciples of Jesus minus Judas?

The 5th prime number which with 13 is the first pair of twin primes (not counting the triplet primes 3 5 7).

Twelve

I'll sing you twelve, O
Green grow the rushes, O
What are your twelve, O?
Twelve for the twelve Apostles



Twelve tribes of Israel and twelve disciples of Jesus

The number of sides in a regular dodecahedron.

The first abundant number whose proper divisors 1, 2, 3, 4, 6 sum to 16 which is greater than 12. All previous number were deficient (except for 6 which is perfect)

There are twelve months in a year

A dozen

Number of People on a Jury

Green Grow the Rushes, O

I'll sing you twelve, O
Green grow the rushes, O
What are your twelve, O?
Twelve for the twelve Apostles
Eleven for the eleven who went to heaven,
Ten for the ten commandments,
Nine for the nine bright shiners,
Eight for the April Rainers.
Seven for the seven stars in the sky,
Six for the six proud walkers,
Five for the symbols at your door,
Four for the Gospel makers,
Three, three, the rivals (arivals?),
Two, two, the lily-white boys,
Clothed all in green, O
One is one and all alone
And evermore shall be so.

The numbers ϕ (Phi) e, (Phaethon's Ride, Euler's Identity) and π (pi) are covered in separate poems.

Question! Why stop here?

What about 13? And why is it considered unlucky?

Finally …

"God created the integers, all else is the work of man."
-- Leopold Kronecker (1823-1891)

Cantor's Ghazal

Abraham looks toward the stars, Albireo, Aldebaran, Altair…past Olber's dilemma,
searching an invisible infinite.
What is beyond number, outside time, space, not dark or light, not substance,
not emptiness, incomprehensible infinite?

Over and again, on the Nile's banks, chaos murders order, Set scatters pieces of Osiris
among grains of desert sand.
Isis journeys to the uttermost parts, sifts and gathers, reconstructs the body,
near-cyclical, countable infinite.

Guide who taught men to know the revolving sky, Atlas picked the wrong side
in Cronos' battle against immortal gods.
Condemned to stand, day on night, bearing our massive celestial sphere
into an achingly tedious, countable infinite.

Zeno's arrow traces through instants or durative allotments, digital/analog
reckoning of the inscrutable clock.
Segments or smooth flight, particle, wave, unbroken arc, parcels parsed into nothing,
cusp of uncountable infinite.

Child Krishna opens his mouth, Yashoda peers down his throat at the universe
and all that is not the universe.
Baby teeth surround the vast, infinitesimal, the real, imaginary, broken, continuous,
countable, uncountable infinite.

-- E R Lutken (3: A Taos Press © 2021)

Counting: 1,2,3, … infinity

The opening chapter (subtitled "How High Can You Count") of Geoge Gamow's (1904 – 1968) book *One, Two Three … Infinity* begins with the following humorous story.

> "There is a story about two Hungarian aristocrats who decide to play a game in which the one who calls the largest number wins.
>
> 'Well,' said one of them, 'you name your number first.'
>
> After a few minutes of hard mental work, the second aristocrat finally named the largest number he could think of.
>
> 'Three,' he said.
>
> Now it was the turn of the first one to do the thinking, but after a quarter of an hour he finally gave up.
>
> 'You've won,' he agreed."

The purpose of this story is not to cast dispersions on Hungarians. Indeed some of the most brilliant mathematicians like John von Neumann, Paul Erdos, and Gorge Polya were Hungarian. The purpose of the story is to introduce the reader into the idea of "how people count".

Undoubtedly the earliest application/use of mathematics was simply to count things which leads to the question of how far can you count? Given any counting number (integer) there is always one more which leads quite naturally to the idea of infinity.

The *potential infinite* is the idea that given any large number N, there is always a larger one say N+1. Compare this to the idea of the *completed infinite*, that somehow you could have a set containing an infinite number of elements!

Abraham looks toward the stars, Albireo, Aldebaran, Altair…past Olber's dilemma,
searching an invisible infinite.

Olbers' Paradox (Heinrich Wilhelm Olbers 1758-1840) or the Dark-Sky Paradox considered the question that if the universe was infinite, homogeneous, and static then anyplace you looked at night you should always see a star since the set of stars was a *completed infinity*. But if this was the case, why was the night sky black?

Abraham

Now consider Abraham's challenge.

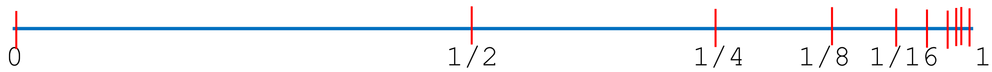
After these things the word of the Lord came to Abram in a vision. "Do not be afraid, Abram, I am your shield; your reward shall be very great." But Abram said, "O Lord God, what will you give me, for I continue childless, and the heir of my house is Eliezer of Damascus". And Abram said, "You have given me no offspring, and so a slave born in my house is to be my heir." But the word of the Lord came to him, "This man shall not be your heir; no one but your very own issue shall be your heir." He brought him outside and said, "*Look toward heaven and count the stars, if you are able to count them.*" Then he said to him, "So shall your descendants be" And he believed the Lord, and the Lord reckoned it to him as righteousness. Genesis 15:1 – 6 NRSV

Zeno's Paradox

Zeno's arrow, one of Zeno's paradoxes, was used to demonstrate the impossibility of motion. For an arrow to fly from point A to point B, it had to cover half the distance first. But then it had to cover half the distance from A to that halfway point, and thence to first cover the distance between A and a previous halfway point etc. etc. etc. In other words, *it had to cover the distance of an infinite number of sub-intervals first. So how can it move at all – it would take an infinite amount of time*.

Thus, motion is impossible.

The way out of this dilemma is our (modern) understanding that *infinite processes can terminate in a finite value* – in this case $\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+...+\frac{1}{2^k}+...=\sum_{k=1}^{\infty}\frac{1}{2^k}=1$. That is half the distance traveled plus half of the half (or a quarter of the distance traveled) plus a half of the half of the half (an eighth) etc. sums to 1 – the entire distance from A to B.



Think of it this way:

$$\frac{1}{2}+\frac{1}{4}=\frac{3}{4}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}=\frac{7}{8}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}=\frac{15}{16}$$
$$\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{16}+\frac{1}{32}=\frac{31}{32}$$

etc…

Do you see a pattern here? The sums keep getting closer and closer to 1 and since I can get as close to 1 as I want (every time I add in a new term, I halve the distance to 1), what's to prevent me from saying that maybe this infinite sum converges to 1. See *Measured Illusion* and the *Completeness Axion for Reals*.

There Be Dragons Here …

Of course, a completely rigorous mathematical solution to Zen's Paradox, that an infinite summation can terminate in a finite value, lay almost 2000 years in Zeno's future!

Cantor and the Countably Infinite

In his book *Dialogues Concerning Two New Sciences* published in 1638, Galileo (1564 – 1642) confronted the problem of the infinite by observing that while on one hand each integer has a square (1^2 = 1, 2^2 = 4, 3^2 = 9, 4^2 = 16 …) which seem to imply there were as many squares as integers, yet on the other hand, there were obviously more integers than squares since not all integers were squares.

Somehow comparisons of less than, equal to, or greater than could not be applied to infinite sets. It was Georg Cantor (1845 – 1918) who unraveled this knot by stating that two sets had *equal cardinality* if and only if they could be put into one-to-one correspondence with each other.

Mathematically speaking, the cardinalities of two sets A and B are equal, written |A| = |B| if and only if there is a one-to-one onto mapping (i.e. function) from A to B; that is f:A → B is one-to-one and onto.

So, in the case of the integers and squares, since you could line them up and match (see below)…

1	2	3	4	5	6	7	8	9	10	…
1	4	9	16	25	36	49	64	81	100	…

.. the two sets had the same *cardinality*. Note that this is also true for finite sets.

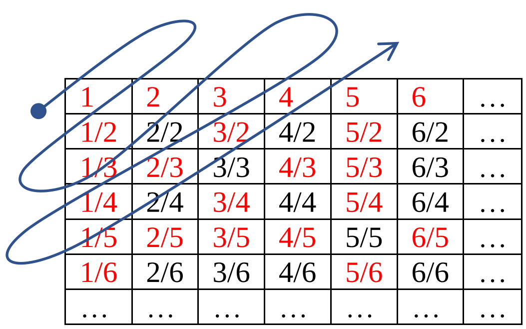
The fact that the set of squares form a *proper subset* of the set of integers expresses a property of infinite sets: an infinite set S can be put into a one-to-one correspondence with a proper (infinite) subset of set S.

Sets which could be put into a 1:1 correspondence with the natural numbers 1,2, 3, … are *countably infinite*. Cantor called this infinity $\aleph_0$, aleph naught (*aleph* being the first letter of the Hebrew alphabet).

Cantor went on to show that the integers, positive and negative were also countably infinite as were the rational numbers. To show this all that was needed was to demonstrate a 1:1 correspondence with the natural numbers. For example …

Natural Numbers:	1	2	3	4	5	6	7	8	9	10	…
Integers:	0	+1	-1	+2	-2	+3	-3	+4	-4	+5	…

The set of positive rational numbers can be enumerated using a square table (see below). Using a zig-zag curve through the table one could pair each rational number with a natural number skipping over any rational number (in black) which had already been paired off (in red).



1	2	3	4	5	6	…
1/2	2/2	3/2	4/2	5/2	6/2	…
1/3	2/3	3/3	4/3	5/3	6/3	…
1/4	2/4	3/4	4/4	5/4	6/4	…
1/5	2/5	3/5	4/5	5/5	6/5	…
1/6	2/6	3/6	4/6	5/6	6/6	…
…	…	…	…	…	…	…

Natural Numbers:	1	2	3	4	5	6	7	8	9	10	…
Rational Numbers:	1	2	1/2	1/3	3	4	3/2	2/3	1/4	1/5	…

Cantor and the Uncountably Infinite

However, when it came to the set of real numbers Cantor proved that no such one-to-one correspondence existed! Since the set of real numbers could not be put into a 1:1 correspondence with the countably infinite set of the natural numbers, cardinality of the set of real numbers was a larger infinity – that is *uncountably infinite*.

Cantor's Diagonalization Proof

Cantor began by considering just a subset of the reals; that is the set of reals on the interval (0,1) Then he assumed the opposite - that there really ***was*** a 1:1 correspondence between the natural numbers and the real numbers on the interval (0,1). He then obtained a contradiction forcing him to conclude that his original assumption that there was a 1:1 correspondence between the natural numbers and the set of reals on the interval (0,1) was not true but *false*.

Real numbers on the interval (0,1) can be written as infinitely long decimal expansion like 0.50000, $0.\overline{333}$ or 0.1415926… etc. So, if there was a 1:1 correspondence between the natural numbers and the real on the interval (0,1) he could list then out something like this …

```
1 | 0.50000 …
2 | 0.33333 …
3 | 0.14159 …
4 | 0.17769 …
5 | 0.13628 …
6 |
```

His diagonalization proof then works as follows. He ran a diagonal down this list of reals and changed the n^{th} digit in the n^{th} number, sort like this

```
1 | 0.60000 …
2 | 0.34333 …
3 | 0.14259 …
4 | 0.17779 …
5 | 0.13629 …
6 |
```

Here we add one to each digit and if the digit was 9 we make it 0.

This new real number 0.64279… differs by one digit from every other digit in the list so it can't be in the list – except we assumed that every real number on the interval (0,1) was in the list!

We have a contradiction, and we are forced to conclude that there is no 1:1 correspondence between the natural numbers and the set of reals on the interval (0,1). Moreover, since it can be easily shown that there is a 1:1correspondence between the set of reals on the interval (0,1) and the set of all the reals, we have proved that the real numbers are not countably infinite but *uncountably infinite*.

Cantor used the letter c (for continuum) to denote the cardinality of the reals. Therefore $\aleph_0 < c$

Some infinites are larger than other infinities.

Cantor went on to construct sets whose cardinality was larger than the cardinality of the reals.

Cantor went on to prove that the cardinality of the power set for a set S, denoted P(S) is greater than the cardinality of S; that is |P(S)| > |S|. The power set P(S) of a set S is the set of all subsets of S. This is easily seen for finite sets.

For example: If $S=\{a,b,c\}$ then $P(S)=\{\varnothing,\{a\},\{b\},\{c\},\{a,b\},\{b,c\},\{a,c\},\{a,b,c\}\}$ is larger.

In fact, for S a finite set, $|P(S)|=2^{|S|}$. For example if the cardinality of S is 3, the cardinality of the power set $P(S)=2^3=8$. Cantor was able to prove this inequality for infinite sets; that is $|S|<|P(S)|$ (although $2^{|S|}$ doesn't make much sense if $|S|$ is infinite). What this means is the that *set of all subsets* of the real numbers has a greater cardinality than the set of all real numbers – an even larger infinity. And of course, why stop there? If we denote $\mathbb{N}$ as the set of natural numbers and $\mathbb{R}$ as the set of all real numbers, then

$$\aleph_0=|\mathbb{N}|<c=|\mathbb{R}|<|P(\mathbb{R})|<|P(P(\mathbb{R}))|<|P(P(P(\mathbb{R})))|<\ldots<|P^n(\mathbb{R})|<\ldots$$

The Continuum Hypothesis

Question! Is there a *proper subset* set S of reals (let $\mathbb{R}$ denote the set of all reals) i.e. $S\subseteq\mathbb{R}$ *and* $S\neq\mathbb{R}$ such that $\aleph_0<|S|<c$? In other words, is it possible that *c* is not the *next largest* infinity after $\aleph_0$? (Aside - If we designate $\aleph_1$ to be the next largest infinity after $\aleph_0$ then we're asking which is true: $\aleph_1<c$ or $\aleph_1=c$?)

The Continuum Hypothesis states that there is no infinite set S whose cardinality is strictly between $\aleph_0$ and c, or to put it another way $c=\aleph_1$. Cantor tried to prove this true and then tried to prove this false. He got nowhere with it (which may have contributed to his eventual mental breakdown).

It turns out that the Continuum Hypotheses is *independent* of the axioms for set theory; that is, given the set of axioms used to prove results (theorems) about sets, the continuum hypothesis can neither be *proved* nor *disproved* : so you can have one *form* of set theory where the Continuum Hypothesis (denoted CH) is assumed true and another *form* of set theory where the Continuum Hypothesis (denoted ¬ CH) is assumed to be false. That is adding either CH or ¬ CH as an additional axiom to the set of axioms for set theory results *in two different set theories* with different results (theorems) proved by each. Take your pick!

Irrational Loss

After she's fed, teeth brushed, shuffled into bed,
he breaks out of the dungeon, desperate for air,
choking on lucid memories of rational years,
easy, hopeful reckonings of what lay ahead,
now nothing but dank clutter, a slew of the unsaid,
unsayable. Her wounded mind never clears –
discussions, odes, lullabies all lost to tears.
He voices it; the wish that she were dead,
as if that would bring closure. Multiply
the irrational by zero, it should disappear –
no – subtract one (void is the deceased's concern).
She won't be dead for him, not even when she dies,
but forged within shackles of a garbled remainder,
stray glints of the smile, the music might return

-- E R Lutken (3: A Taos Press © 2021)

The Problem of Irrational Numbers

The ancient Greeks (e.g. the Pythagorean brotherhood) initially believed that all numbers were *commensurate*, that is, given any two numbers x and y there was a third number z and integers n and m such that

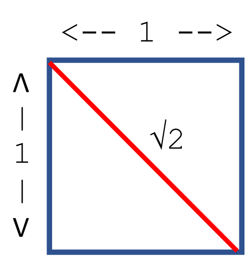
$$x = n \times z \text{ and } y = m \times z$$

That is, both x and y were *integer multiples* of a common number z.

Today we'd say that all numbers were *rational numbers*; that is, any number c is expressible in the form $\frac{a}{b}$ where a and b are integers and $b \neq 0$ (since you cannot divide by zero).

However, the ancient Greeks discovered that this was not always the case. Some numbers like $\sqrt{2}$ were not *commensurate* with any integer. Today we say $\sqrt{2}$ is an *irrational* number.

They were wrong – and they knew it!



"perfect square's spine"

This discovery is sometimes referred to as the *first logical scandal in mathematics* (recall *Fundamentals of Mathematics*).

It's rather easy to prove that $\sqrt{2}$ is not rational; that is there are no two integers *a* and *b* ($b \neq 0$) such that $\sqrt{2} = \frac{a}{b}$. The method used is called an *indirect proof* (aka *reductio ad absurdum*) where an assumed hypotheses logically leads to a blatant contradiction (the technique used in *Cantor's Ghazal*). Therefore, the assumed hypothesis must be false.

To begin: Assume that there *are* integers a and b such that $\sqrt{2} = \frac{a}{b}$ and that *a* and *b* have no common factors (otherwise factor them out) - an easy hypothesis and important as we shall see.

So $\sqrt{2} = \frac{a}{b}$ yields $\sqrt{2} \cdot b = a$ and squaring both sides gives you $2 \cdot b^2 = a^2$ so a^2 is an even integer (since 2 is one of it factors or to put is another way, an i*nteger n is even* if and only if $n = 2 \cdot k$ for some integer *k*).

However, if a^2 is even, then so is *a*. This is easy to prove because if *a* were *odd*, that is $a = 2k+1$ for some integer *k*, then $a^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1$ which means a^2 is odd (which it's not!).

Now since a must be *even*, that is $a = 2k$ for some integer *k*, $a^2 = 4k^2$ so $2b^2 = a^2 = 4k^2$ or $b^2 = 2k^2$ so b^2 is even implying that *b* is also even!

But wait! If both *a* and *b* are even then they would have a common factor 2, which contradicts the original hypothesis that *a* and *b* have no common factors!!!

Therefore $\sqrt{2}$ is not rational. QED!

So, we define the set of all numbers which are NOT rational as the set of *irrational numbers* .

Together the union of the disjoint sets of the *rational numbers* and the *irrational numbers* make up the set of all *real numbers*.

Since it's difficult to define something as not having a property (an irrational number is one which is NOT a rational number) there is another way to define an irrational number.

Rational number have the property that their decimal expansions will eventually repeat, and this repetition can be used to express the number in the form of a/b where a and b are integers with no common factors.

Example: Convert the repeated decimal 0.17291729…into the rational form a/b where integers a and b have no common factors. To begin let $S = 0.1729\overline{1729}$ If we multiply S by 10,000 we have $10{,}000S = 1729.1729\overline{1729}$. Now subtract $S = 0.1729\overline{1729}$ from $10{,}000S = 1729.1729\overline{1729}$ to obtain $9999S = 1729$ or $S = \frac{1729}{9999} = \frac{7 \times 13 \times 19}{3 \times 3 \times 11 \times 101}$, a rational number (no common factors)

Therefore, we can also define an irrational number is a number whose decimal expansion *never repeats*.

A Thought Experiment: If you had a fair ten-sided die and rolled it an *infinite number* of times (?) writing down each digit which came up, you would have generated an irrational number - assuming that the sequence of random digits so generated cannot repeat. Of course, stopping at any point would make the number rational.

The Misfortune of Hippasus

Legend has it that the existence of irrational quantities was a closely held (and embarrassing) secret of the Pythagorean Brotherhood and the person who revealed the secret, Hippasus, was tossed overboard from a ship (to drown?).

After she's fed, teeth brushed, shuffled into bed,
he breaks out of the dungeon, desperate for air,
choking on lucid memories of rational years,
easy, hopeful reckonings of what lay ahead,
now nothing but dank clutter, a slew of the unsaid,
unsayable

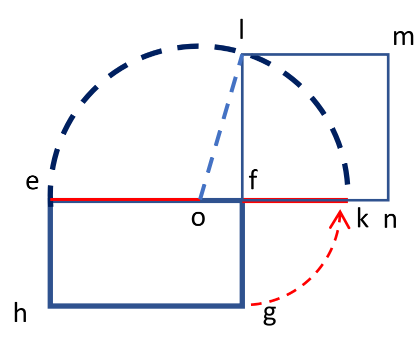
The Vast Numerical Realm beyond the Rational Numbers

As mentioned in *The Fundamentals of Mathematics* and *Cantor's Ghazal*, Georg Cantor (1845 - 1918) proved that the rational numbers were *countably infinite*, that is there was a one-to-one correspondence between the natural numbers 1,2,3 … etc. and the rational numbers but no one-to-one correspondence could be established between the set of all real numbers and the set of natural numbers; the real numbers were *uncountably infinite*. Thus, in some hard well defined mathematical sense there are more real numbers (which include the rational numbers as a proper subset) then the set of rational numbers. The difference is the set of irrational numbers which must be *uncountable infinite*.

However, between (including) the set of rational numbers and the set of all real numbers there are a number of nested sets in-between which contain the rational numbers but also include some subset of irrational numbers .

Constructable Numbers

Briefly, a *constructable number* is a quantity that can be constructed using only straight-edge and compass. Based on Euclid's *Elements* (300 BCE), given two lines of length *a* and *b*, using only a straight-end and compass one can construct a third line equal in length to $a+b$, a-b, $a\times b$, a/b and $\sqrt{a}$ (see below). Thus, starting with a unit length, one can construct any rational quantity plus some irrational quantities involving square roots (or fourth roots or eighth roots etc. but 3rd roots are out!). The constructable numbers are an obvious superset of the rational numbers but do not include many irrational numbers (e.g. the *cube root* $\sqrt[3]{2}$)



Construction for $\sqrt{a}$

Let $\overline{ef} = a$ and $\overline{fg} = 1$ so rectangle $\square efgh$ has area a.

Extend $\overline{ef}$ out to k so that $\overline{fk} = \overline{fg} = 1$ and let o bisect $\overline{ek}$. Construct a semicircle with center o and radius

$\overline{eo} = \overline{ko} = \dfrac{a+1}{2}$. Extend $\overline{gf}$ to intersect the semicircle at l and draw $\overline{fl}$. Triangle Δofl is a right triangle with hypotenuse $\overline{ol} = \dfrac{a+1}{2}$ and side $\overline{of} = \dfrac{a+1}{2} - 1 = \dfrac{a-1}{2}$.

By the Pythagorean theorem $\overline{fl} = \sqrt{\left(\dfrac{a+1}{2}\right)^2 - \left(\dfrac{a-1}{2}\right)^2} = \sqrt{a}$. Using $\overline{fl}$ as one side, construct the square $\square\, flmn$ which has area a, same as rectangle $\square efgh$.

Thus, any number that can be expressed using the four arithmetic operations of addition, subtraction, multiplication and division plus square roots is a constructable number.

Obviously $\sqrt{2}$ is constructable as is the fourth root $\sqrt[4]{2}$ (square root of a square root) as is the golden ratio $\varphi = \dfrac{1+\sqrt{5}}{2} \approx 1.618033989$, irrational and constructable.

Aside from the set of *Constructable Numbers* which include the rational numbers and extend into (include members of) the irrational numbers there are other well-defined subsets of real numbers which also extend into the realm of the irrational numbers: Algebraic, Transcendental, Computable … (see *Meditation on Transcendental Numbers*)

Meditation on Transcendental Numbers

they just keep on lookin' to the east
– Tom Johnston

Falcons stoop towards shape-shifting starling clouds.
Wolves leap into rushing streams of lemmings.
Arithmeticians part points on the endless line
in search of strange, shrouded prey
hovering close, but seldom caught.
Captives refuse to reveal exactly
what they are, why they act as they do,
trails of their jumbled decimals
escaping to hazy infinity.
Not rational, not algebraic,
not $\sqrt{2}$, not φ.

नेति नेति
Neti Neti
Not this, not that.

Cryptic figures teeter between smooth and convoluted,
famous names, π, e, Liouville, Champernowne,
blazing with intricate facets like fiery diamonds,
others resting, placid, obscure as dust.
But all are known by what they are
not:
not Vedic Chant, not YouTube Yoga,
not Doobie Brothers, not Atman.

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Not this, not that.

-- E R Lutken (3: A Taos Press © 2021)

From Counting to Transcendental Numbers

(Continued from *Irrational Loss*)

There are different *kinds* of numbers. It's not that when humans started doing mathematics that we had a firm grasp of number – our idea of what is a number evolved over time as we pushed the boundaries of mathematical knowledge.

One way to classify numbers is to consider what kinds of algebraic operations can be done with them which introduces the idea of *closure*. An operation like addition is *closed* if when applied to two numbers, the result is the same type of number. For example, if you add two positive integers, the result is a positive integer; the positive integers are *closed under addition*. However, if you subtract one positive integer from another, say 5 – 7, the result might not be a positive integer; the positive integers are *not closed* under subtraction.

With this in mind, we can use closure to identify nested sets or type of numbers: the Natural numbers, the Integers, the Rational numbers, the Real numbers and the Complex numbers using the six algebraic operations, addition, subtraction, multiplication, division, exponentiation and roots (e.g. square roots, cube roots, etc. – the inverse operations of exponentiation)

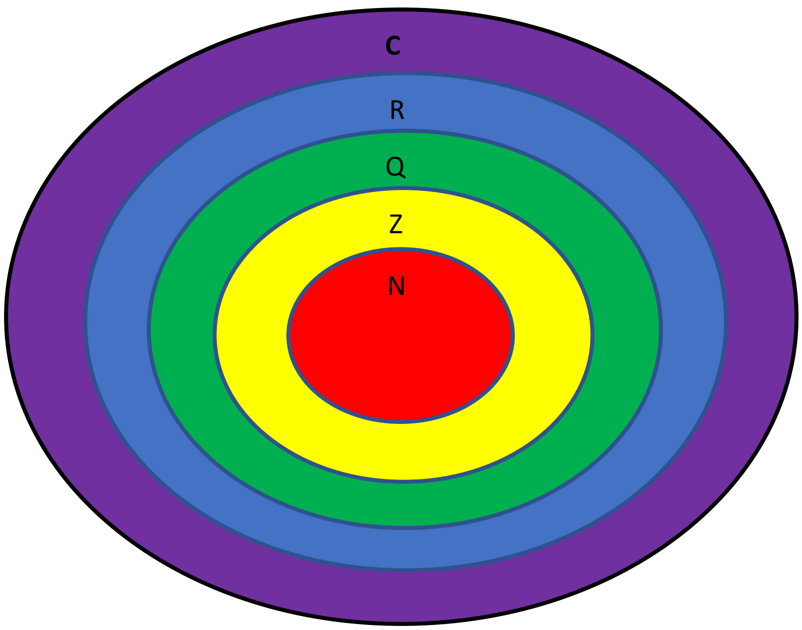
Nine Zulu Queens Rule China – A Nested Venn Diagram

N: The Natural (or Counting) numbers: 1, 2, 3, 4, 5, …

The Natural numbers are *closed* under addition and multiplication meaning that if you add or multiply two natural numbers, the result is a natural number. However, the natural numbers are not *closed* under subtraction (the inverse of addition). 0 may or may not be considered to be a Natural number (strictly speaking it's not).

Z (German – Zahlen to count): The Integers positive and negative …, -3, -2, -1, 0, 1, 2, 3, …

The Integers which include the Natural numbers are *closed* under subtraction (as well as addition and multiplication) but they are not closed under division (the inverse of multiplication)



A Venn Diagram of Nested Sets of Numbers - $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$

It's interesting that even as late as the 16th century in Europe, negative numbers were not considered to be *numbers* per say.

Q (Quotient): The Rational Numbers, numbers of the form a/b where a and b are integers and $b \neq 0$.

The Rational Numbers which include the Integers are *closed* under addition, subtraction, multiplication, division and exponentiation. But what about square roots or nth roots (i.e. $\sqrt[n]{a}$ for integer n and a > 0), the inverse of exponentiation etc.? Historically Rational Numbers (fractions) were used and accepted before negative numbers.

R: The Real numbers fill in the gaps between the rational numbers. For example, $\sqrt{2}$ is not a rational number (recall *Irrational Loss*).

The Real numbers fill in the *gaps* between Rational Numbers. They allow the taking of square roots, cube roots and nth roots of positive reals. But what about square roots or nth roots (for n even) of negative reals?

C: The Complex Numbers where $i = \sqrt{-1}$. Complex numbers are of the form $a + bi$ where a and b are Real numbers. For example, $-\frac{1}{2} + \frac{\sqrt{3}}{2}\imath$ is one of the cube roots of 1.

The Complex numbers are *closed* under addition, subtraction, multiplication, exponentiation and nth roots. This completes the closure of the numbers for all six algebraic operations. See *Euler's Identity*.

The Irrationals: Constructable, Algebraic, Transcendental

The irrational numbers (see *Irrational Loss*) are the numbers which are not rational- a negative definition – defining something as not having some property. The Irrational Numbers are the Real numbers minus the Rational numbers: $\mathbb{R} - \mathbb{Q}$.

Constructable Numbers are numbers that can be constructed using a straight-edge and compass; they are geometrically based. Algebraically speaking, any expression involving the four standard operations of addition, subtraction, multiplication, and division plus squaring and square roots is a constructable number (see *Irrational Loss*).

For example: $\varphi = \frac{1+\sqrt{5}}{2}$

But there is more

Arithmeticians part points on the endless line
in search of strange, shrouded prey
hovering close, but seldom caught.
Captives refuse to reveal exactly
what they are, why they act as they do,
trails of their jumbled decimals
escaping to hazy infinity.
Not rational, not algebraic,
not $\sqrt{2}$, not φ.

Neti Neti
Not this, not that.

Algebraic Numbers An algebraic number is a number which is the root of a polynomial with integer coefficients. For example, any rational number $\frac{a}{b}$ is an algebraic number since $\frac{a}{b}$ is a solution to the linear equation $b \cdot x - a = 0$.

The irrational $\sqrt{2}$ is also an algebraic number since $\sqrt{2}$ is a solution (root) to the quadratic equation $x^2 - 2 = 0$. $\sqrt[3]{2}$ which is not constructable is also algebraic as it's a solution to the cubic $x^3 - 2 = 0$. In fact, any nth root $\sqrt[n]{a}$ for integer n is an algebraic number since $\sqrt[n]{a}$ is a solution to $x^n - a = 0$. The set of Constructable Numbers form a *proper subset* of the set of Algebraic Numbers.

Moreover, the set of Algebraic Numbers (and the set of Constructable Numbers) are also *countably infinite* (recall *Cantor's Ghazal*)! That is a one-to-one correspondence can be formed between the set of Algebraic Numbers and the set of Natural Numbers.

Transcendental Numbers: Numbers which are not Algebraic Numbers are called Transcendental Numbers. Thus, the two mutually exclusive sets of Algebraic Numbers and Transcendental Numbers make up the *uncountably infinite* set of real numbers.

What is amazing is that the first transcendental number wasn't found until the 19th century. Later on in the 19th it was discovered (proved) that both π and e were both transcendental numbers.

"Cryptic figures teeter between smooth and convoluted,
famous names, π, e, Liouville, Champernowne,
blazing with intricate facets like fiery diamonds,
others resting, placid, obscure as dust.
But all are known by what they are
not:
not Vedic Chant, not YouTube Yoga,
not Doobie Brothers, not Atman.

नेति नेति
Neti Neti
Not this, not that."

Transcendental numbers are very hard to find!

Leonard Euler (1707 – 1783) first described or defined a transcendental number as a number which was ***not*** the roots of a polynomial with integer coefficients. As mentioned above none were found (i.e. proved to be transcendental) until the next century.

Joseph Liouville (1809-1882) in 1851 proved that the number

$$\sum_{k=1}^{\infty} \frac{1}{10^{k!}} = \frac{1}{10} + \frac{1}{10^2} + \frac{1}{10^6} + \frac{1}{10^{24}} + \ldots + \frac{1}{10^{k!}} + .. = 0.110001000000000000000000100..$$

was transcendental. Note that the powers of 10 are raised to *increasing-factorial* values.

Charles Hermite (1828-1901) in 1873 proved $e=\sum_{k=0}^{\infty}\frac{1}{k!}=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...+\frac{1}{k!}+...$ was transcendental.

Ferdinand Lindemann (1852-1939) in 1882 proved π was transcendental The fact that π was proved transcendental put the rest a 2000-year-old quest to *square the circle*; that is given a square and using only straight-edge and compass, construct a circle whose area was the same (i.e. prove π is a *constructable number*). Since π is *transcendental* and therefore NOT *constructable*, you cannot square the circle.

David Champernowne (1912-2000) described in 1933 the Champernowne constant which was later shown to be transcendental.

0.123456789101112131415161718192021….979899100101…

Do you see the pattern?

In 1934 A.O. Gelfond and T. Schneider proved that if $\alpha \neq 0,1$ is any *algebraic* number and β is any *irrational* number then α^{β}, for example $2^{\sqrt{2}}$, is *transcendental*. See ***Ode to*** $2^{\sqrt{2}}$

And then there was Georg Cantor and his infinities (see Cantor's Ghazal)

Recall from *Cantor's Ghazal* that Georg Cantor (1845 – 1918) proved in 1874 that the real numbers could not be put into a one-to-one correspondence with the integers thereby establishing in some strong mathematical sense that there were different sizes of infinite sets and that the size (or cardinality) of the set of integers, called countably infinite, was smaller than the size (or cardinality) of the set of the uncountably infinite real numbers.

He also showed that like the integers, the set of rational numbers and *algebraic numbers* were also countably infinite and so if you remove the countable set of algebraic numbers from the uncountable set of reals, what remains is the *uncountable set of transcendental numbers.*

Therefore, in some well-defined mathematical sense, there are more transcendental numbers than algebraic numbers (which include the rational and integers) and yet as mentioned above the former are hard to find!

To put it poetically: "*The algebraic numbers are spotted over the plane like the stars against a black sky; the dense blackness is the firmament of the transcendentals*" – E.T. Bell

Enter Alan Turing (see Metempsychosis).

One of Alan Turing's (1912-1954) results was to define, or better put, suggest a definition of what it means to compute which in turn leads to the idea of a *computable number*. Basically, a number is computable if there is a Turing Machine (equivalent to a modern computer) or a computer program which can compute the number. That is, a *number is computable* if there is a Turing Machine that can generate it or in the case of an irrational number like π, a Turning Machine that can produce its digits to any degree of accuracy. It's not difficult to see that any algebraic number is computable since the techniques to find the zeros of any polynomial to any degree of accuracy are well known and so is easily (?) programmed. So, all algebraic numbers are computable and well as many transcendental numbers like π and e.

However, Turing showed that there are only countably many Turing Machines (see *Metempsychosis*). Thus, given the fact that there are uncountably many transcendental numbers, most of them, an uncountable number of them, cannot be computed and are therefore in some sense forever unknowable, beyond the pale of human knowledge.

The Set of Computable Numbers against …

Since the set of Turing Machines (think computer programs) is only countably infinite, the set of Computable Numbers is also only countably infinite while …..

… the Vast Realm of the Unknowable

… the remaining uncountably infinite set of Transcendental Numbers are not computable – thus unknowable –undiscoverable?

Falcons stoop towards shape-shifting starling clouds.
Wolves leap into rushing streams of lemmings.
Arithmeticians part points on the endless line
in search of strange, shrouded prey
hovering close, but seldom caught.
Captives refuse to reveal exactly
what they are, why they act as they do,
trails of their jumbled decimals
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Not rational, not algebraic,
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Cryptic figures teeter between smooth and convoluted,
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Ode to $2^{\sqrt{2}}$

Gelfond-Schneider constant, Hilbert number

2:
earth and sky
day and night
bone of bone
flesh of flesh
four arms, legs
double-faced unions
hopping into the arc
from every corner
sing “O soave fanciulla”
dancing with the stars

√2:
brazen scar across harmony
perfect square’s spine
cleaved to the root
Hippasus of Metapontum
murdered for unveiling
that maiden glimpse of madness

$2^{\sqrt{2}}$:
the only even prime
raised to the power of a number
already loosed from the rational
comes uncoupled from the algebraic
the newborn expression
moves, shadowless
towards unfettered infinity

-- E R Lutken (3: A Taos Press © 2021)

2 to $\sqrt{2}$ to $2^{\sqrt{2}}$: an interesting progression – rational to irrational/algebraic to transcendental

2

Rational Numbers

2 is an integer – a whole number. And as the poem suggests, pairs, partners, opposites, and duals seem to be omnipresent in our world. Two doubles into four. Marchs and foxtrots – 2/4/ and 4/4 time.

On/off, hi/lo, 1/0 – binary numbers - base 2 is important for computers. Why? Because it’s easy and cheap to build computers out of bi-stable electronic components.

Binary addition is easy: $0 + 0 = 0$, $0 + 1 = 1$, $1 + 0 = 1$, $1+ 1 = 10$

Binary multiplication is easy: $0 \times 0 = 0$, $0 \times 1 = 0$, $1 \times 0 = 0$, $1 \times 1 = 1$

In fact, all operations including multiplication, division, and even taking square roots, are easy to compute using binary notation. The only drawback is you need lots of 0's and 1's to represent even a fairly small number. Recall *-2*

√2

Irrational and Algebraic Numbers

Recall from *Irrational Loss* that the Pythagoreans, early Greek mathematicians, believed that all numbers were commensurate – that is given any two numbers x and y there was a third number z and integers n and m such that

$$x = n \times z \text{ and } y = m \times z$$

That is, both x and y were *integer multiples* of a common number z.

Today we'd say that all numbers were rational; that is, for any number x, there are integers a and b where b does not equal zero such that $x = \frac{a}{b}$

Recalling the proof presented in *Irrational Loss*, a demonstration that √2 is not rational is rather simple – we assume that opposite and derive a contradiction!

√2 cannot be rational

Recall the legend of the Greek Hippasus who revealed that $\sqrt{2}$ was not rational. For revealing the truth, he was tossed overboard a ship to drown.

Recall from *Meditation on Transcendental Numbers* an *algebraic* number is one which is the root of a polynomial equation with integers coefficients. For example, √2 is *algebraic* since it is the root of the polynomial equation $x^2 - 2 = 0$.

In general, if $p_n(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_2 x^2 + a_1 x + a_0$ is a n^th^ degree polynomial where all coefficients a_k for $0 \le k \le n$ are integers, then α is an *algebraic* number if $p_n(\alpha) = 0$.

$$2^{\sqrt{2}}$$

Pushing the envelope – non-algebraic or transcendental numbers.

Numbers which are not, *algebraic*, not the roots of polynomial equations with integer coefficients, are called *transcendental* numbers – sort of like *irrational* numbers are numbers which are not *rational* numbers. See "*Meditation on Transcendental Numbers*"

It's difficult to define something which is *not something* – which is why *transcendental* numbers are so hard to find. In fact, while transcendental numbers were defined in the 18th century by Euler, it wasn't until the second half of the 19th century that one was actually found, which brings us to the 20th century and to David Hilbert's Famous 23 Problems from his address made at the International Congress of Mathematicians in 1900.

Hilbert's 7th Problem (as addressed to the congress) asked to establish the transcendence of certain numbers – in this case $2^{\sqrt{2}}$. Could it be shown that $2^{\sqrt{2}}$ was *transcendental* (as well as other numbers in this form where a rational number was raised to an irrational power)?

We note in passing that $\sqrt{2}^2 = 2$ is a *rational* number.

In 1934 A.O. Gelfond and T. Schneider proved that if $\alpha \neq 0,1$ is any *algebraic* number and β is any *irrational* number then α^{β}, for example $2^{\sqrt{2}}$, is *transcendental*.

Hence the Gelfond-Schneider *transcendental* constant $2^{\sqrt{2}}$

> "the only even prime
> raised to the power of a number
> already loosed from the rational
> comes uncoupled from the algebraic"

One nice fallout of this result is there is a way to generate a lot of transcendental numbers, for example

$$\{2^{\sqrt{n}} \mid n \text{ is not a square integer}\} \text{ or } \{n^{\sqrt{2}} \text{ where } n = 2,3,4,\ldots\}$$

Question! How does one raise an integer to an irrational power?

First, every irrational number can be seen as the limit of a converging sequence of rational numbers.

$$\text{For example, } \sqrt{2} \approx 1.4142 = 1 + \frac{4}{10} + \frac{1}{100} + \frac{4}{1000} + \frac{2}{10,000}$$

Moreover if a/b is a rational number (both a and b are integers), then any number raised to a rational exponent can be expressed as an integer root of an integer power. For example: $2^{a/b} = \sqrt[b]{2^a}$.

Putting this all together we can approximate $2^{\sqrt{2}}$ to any degree of accuracy (it's computable – see *Metempsychosis*) so

$$\sqrt{2} \approx 1.4142 = 1 + \frac{4}{10} + \frac{1}{100} + \frac{4}{1000} + \frac{2}{10,000}.$$

Therefore $2^{\sqrt{2}} \approx 2^{1.4142} = 2^{1+\frac{4}{10}+\frac{1}{100}+\frac{4}{1000}+\frac{2}{10,000}} = 2 \cdot 2^{\frac{4}{10}} \cdot 2^{\frac{1}{100}} \cdot 2^{\frac{4}{1000}} \cdot 2^{\frac{2}{10,000}} =$
$2 \cdot \sqrt[10]{2^4} \cdot \sqrt[100]{2^1} \cdot \sqrt[1000]{2^4} \cdot \sqrt[10000]{2^2} \approx 2.665119089$

… as computed on a hand held (TI-84) calculator.

Metempsychosis

The Turing Test

Offspring of golden Galatea, Helena, Rachael,
these slick new machines dressed in plastic flesh,
ersatz immortals, built with daedal
chips, spit out fabricated tête-à-têtes
aimed at pudgy analogs who stare at blue screens'
handsome would-be reflections, perfect designs
pressed from juicers pouring out easy streams
of pathos gleaned from a million lonely minds,
all for stamped approval on their integrated circuits.
Hell, we talk to our cars, why not robots?
The test: if we can't distinguish that
they are not us, they must be us.
But do they give a shit about the sky?
They say they do

--E R Lutken (3: A Taos Press © 2021)

Alan Mathison Turing (June 23, 1912 – June 7, 1954)

Alan Turing is known for many things but a few of the more important are given below.

A graduate of Kings College, Cambridge University with a degree in mathematics Turing came to be known for the 1936 publication of his paper titled *On Computable Numbers, with an application to the Entscheidungsproblem*, the solution to the so-called decision problem in mathematics. That is, is there a *decision algorithm* that can decide whether any statement in mathematics is true or false. His paper proved the answer was no!

The difficulty with this problem is that of finding a definition of algorithm strong enough to prove that a *decision algorithm* was impossible. Turing solved this problem by defining a mechanical method, a Turing machine (more below), that could be used to compute answers for mathematical and logical questions.

For example, given any integer n, is n prime? A simple algorithm to answer this questions is to try dividing n by integers starting at 2 and increasing by 1 until the either there is no reminder after a trial division (so n is not prime) or you exceed $\sqrt{n}$ (a standard mathematical result that states that if n is not prime it has a divisor less than or equal its square root) which means n is prime.

During World War II Turing worked at the Government Code and Cipher School Bletchley Park. His work was instrumental in breaking the German Enigma codes.

After the war Turing was invited by the National Physics Laboratory in London to help design a computer called the ACE (Automatic Computing Engine). He subsequently left the NPL to work on the Small-

Scale Experimental Machine, the first working stored-program computer, at the University of Manchester.

Turing's life ended tragically. In 1952 convicted by the British homosexual statues, he opted for chemical sterilization in lieu of prison time. Although stripped of his security clearance he remained academically active. In 1954 he died of potassium cyanide poisoning found in apple next to his bed. Whether his death was suicide (the official cause of his death) or an accident, the mystery remains to this day.

As we'll see below Turing is known for four things.

1. Turing Machines

A mentioned above, Turing invented a simple device capable of computing answers to mathematical questions.

A Turing machine consists of

1. An unbounded two-way tape partitioned into discrete cells that can hold a symbol (like a letter or digit)

2. A read/write head that can …

   read the current symbol,
   write another symbol, and
   then move either one cell to the left or right on the tape.

3. A finite set of current states which with the current symbol under the read/write head determines the next move of the Turing Machine.

Graphically this can be seen as follows …

```
----+---+---+---+---+---+---+--
 .. |   | 1 | 1 | 1 | 0 |   | …      ← Two Way Unbounded Tape
----+---+---+---+---+---+---+--
          ↑A                        ← Read/Write Head positioned on tape cell
                                      Current State (A)
```

A *program* on a Turing machine works as follows. Given the current state and the current tape symbol, the Turing Machine consults a finite table for the next state and next symbol (overwriting the current symbol) then moves the R/W head either to the left or right. A Turing Machine is a *finite state machine*.

A Turing machine program can be given as a set or table of quintuples of the form …

(current state, current symbol, next symbol, next state, L/R)

That is given the current state and current tape symbol, write the next symbol (overwriting the current symbol), advance to the next state, and move the read/write head Left or Right .

Example The following is a seven quintuple Turing Machine Program to add two integers (for example 3+2) in unary notation where an integer *n* is represented by *n* 1's followed by a terminating 0. For example, 5 is represented by 111110 . To add 3 plus 2, start with 1110110 on the tape with the read/write head positioned on the leading 1.

`(A,1,1,A,R)`	on 1 move right – stay in state A
`(A,0,1,B,R)`	found a 0 - convert 0 to 1 - go right – state B
`(B,1,1,B,R)`	on 1 move right stay in state B
`(B,0, ,C,L)`	found 0 - erase 0 - go left – state C
`(C,1,0,D,L)`	convert 1 to 0 – go left - state D
`(D,1,1,D,L)`	on 1 scan left – stay in state D
`(D, , ,E,R)`	on blank go right – state E -halt

```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 | 1 | 0 |   |
--+---+---+---+---+---+---+---+---+--
        ↑A - Repeat 3 times
```

BEGIN: `(A,1,1,A,R)` on 1 move right

```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 | 1 | 0 |   |
--+---+---+---+---+---+---+---+---+--
                    ↑A →B
```

`(A,0,1,B,R)` convert 0 to 1 – state B - go right

```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 1 | 1 | 1 | 0 |   |
--+---+---+---+---+---+---+---+---+--
                        ↑B - Repeat Twice
```

`(B,1,1,B,R)` on 1 move right

```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 1 | 1 | 1 | 0 |   |
--+---+---+---+---+---+---+---+---+--
                                ↑B →C
```

`(B,0, ,C,L)` erase 0 – state C - go left

```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 1 | 1 | 1 |   |   |
--+---+---+---+---+---+---+---+---+--
                            ↑C →D
```

`(C,1,0,D,L)` convert 1 to 0 – state D - go left

```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 1 | 1 | 0 |   |   |
--+---+---+---+---+---+---+---+---+--
                        ↑D - Repeat 5 times
```

`(D,1,1,D,L)` on 1 scan left

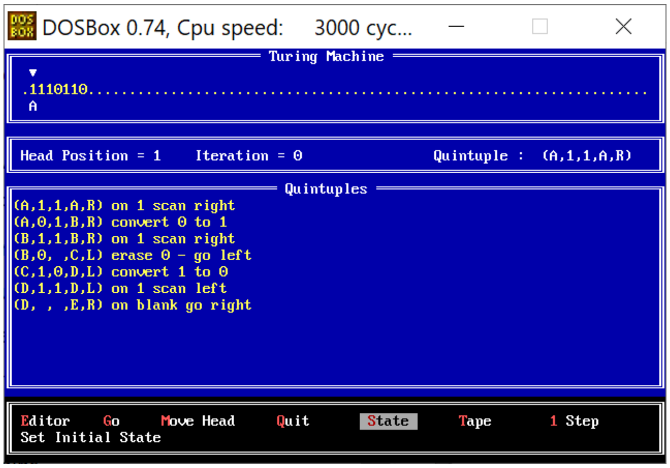
```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 1 | 1 | 0 |   |   |
--+---+---+---+---+---+---+---+---+--
    ↑D →E
```

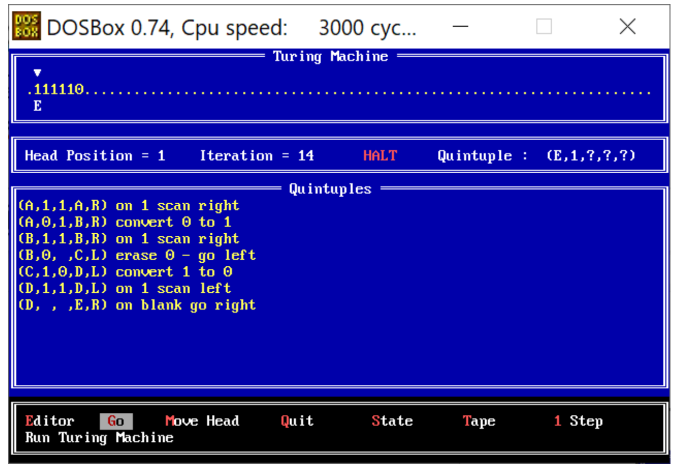
`(D, , ,E,R)` on blank - state E - go right

```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 1 | 1 | 0 |   |   |
--+---+---+---+---+---+---+---+---+--
        ↑E
```

`HALT`: since there is no quintuple for state E

Thus a Turing Machine that can add two integers in unary notation!





Before and after shots of a run of the Turing Machine Adder program run on a Turing Machine simulator

The action of a Turing Machine is completely determined by the current state and the current symbol under the read/write head. Note that the Turing Machine halts if no quintuple exists seen as `(E,1,?,?,?)` for a current state/current symbol configuration.

Example: A Copier Turing Machine (13 quintuples) can duplicate a number. Starting with 1110 in the tape it ends with 11101110.

```
(A,1,x,B,R)   cross off 1 go right and state B
(A,0,0,D,R)   found 0 end of number go right and state D
(B,1,1,B,R)   move right
(B,0,0,B,R)   move right
(B, ,1,C,L)   found blank, write 1 go left
(C,1,1,C,L)   move left
(C,0,0,C,L)   move left
(C,x,1,A,R)   restore crossed off 1 go to state A - repeat
(D,1,1,D,R)   move right
(D, ,0,E,L)   found end - deposit 0 go left
(E,1,1,E,L)   move left
(E,0,0,E,L)   move left
(E, , ,F,R)   found left blank go right - halt
```

```
--+---+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 |   |   |   |   |
--+---+---+---+---+---+---+---+---+---+--
          ↑A  →B
```

`(A,1,x,B,R)` – cross off 1 – state B - go right

```
--+---+---+---+---+---+---+---+---+---+--
  |   | x | 1 | 1 | 0 |   |   |   |   |
--+---+---+---+---+---+---+---+---+---+--
              ↑B    repeat twice
```

`(B,1,1,B,R)` – scan right to 0

```
--+---+---+---+---+---+---+---+---+---+--
  |   | x | 1 | 1 | 0 |   |   |   |   |
--+---+---+---+---+---+---+---+---+---+--
                      ↑B
```

`(B,0,0,B,R)` – skip over 0

```
--+---+---+---+---+---+---+---+---+---+--
  |   | x | 1 | 1 | 0 |   |   |   |   |
--+---+---+---+---+---+---+---+---+---+--
                       ↑B  →C
```

(B, ,1,C,L) – convert blank to 1 – state C - go left

```
--+---+---+---+---+---+---+---+---+---+--
  |   | x | 1 | 1 | 0 | 1 |   |   |   |
--+---+---+---+---+---+---+---+---+---+--
               ↑C
```

(C,0,0,C,L) – skip over 0

```
--+---+---+---+---+---+---+---+---+---+--
  |   | x | 1 | 1 | 0 | 1 |   |   |   |
--+---+---+---+---+---+---+---+---+---+--
           ↑C
```

(C,1,1,C,L) – skip over 1's

```
--+---+---+---+---+---+---+---+---+---+--
  |   | x | 1 | 1 | 0 | 1 |   |   |   |
--+---+---+---+---+---+---+---+---+---+--
       ↑C  →A
```

(C,x,1,A,R) – found x, reset to 1- state A - go right

```
--+---+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 |   |   |   |
--+---+---+---+---+---+---+---+---+---+--
           ↑A
```

(A,1,x,B,R) – repeat with next 1 - cross off a – state B

```
--+---+---+---+---+---+---+---+---+---+--
  |   | 1 | x | 1 | 0 | 1 |   |   |   |
--+---+---+---+---+---+---+---+---+---+--
               ↑B
```

(B,1,1,B,R) -

At this point the read/write head sweeps back and forth two more times placing two more 1's at the end

```
--+---+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | x | 0 | 1 | 1 | 1 |   |
--+---+---+---+---+---+---+---+---+---+--
               ↑C  →A
```

(C,x,1,A,R) – found x, reset t 1 – state A go right

```
--+---+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 | 1 | 1 |   |
--+---+---+---+---+---+---+---+---+---+--
                   ↑A
```

(A,0,0,D,R) – time to finish up! - state D go right

```
--+---+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 | 1 | 1 |   |
--+---+---+---+---+---+---+---+---+---+--
                       ↑D   repeat three times
```

(D,1,1,D,R) – scan right to end

```
--+---+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 | 1 | 1 |   |
--+---+---+---+---+---+---+---+---+---+--
                                   ↑D  →E
```

(D, ,0,E,L) – found end, place 0 – state E go left

```
--+---+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
--+---+---+---+---+---+---+---+---+---+--
                               ↑E
```

(E,1,1,E,L) – scan all the way left

```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
--+---+---+---+---+---+---+---+---+--
         ↑E  repeat six times
```

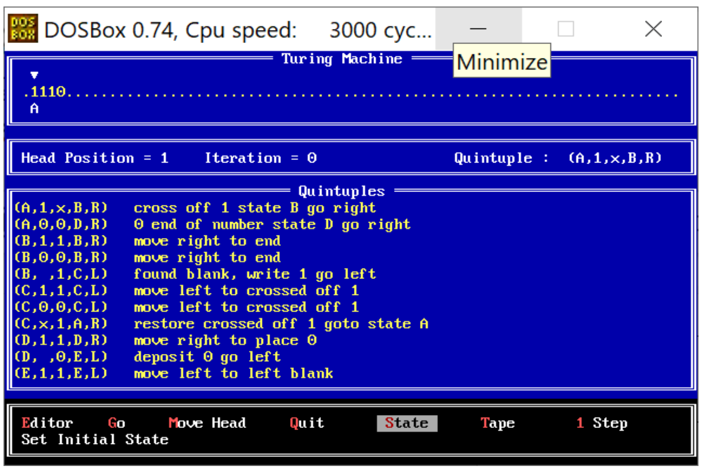
(E,1,1,E,L) and (E,0,0,E,L)
skip over 0's and 1's

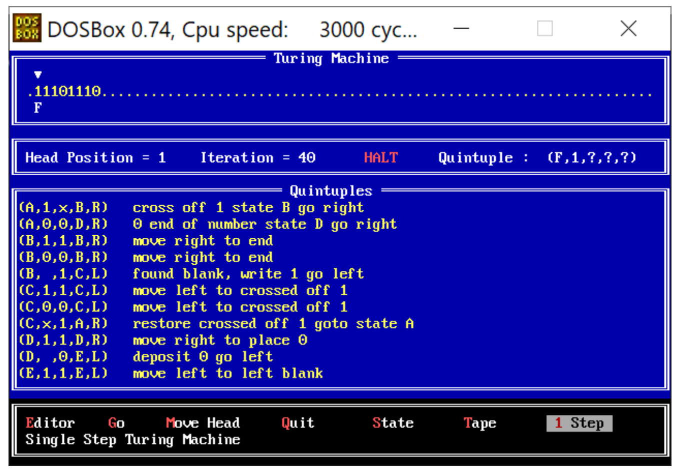
```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
--+---+---+---+---+---+---+---+---+--
     ↑E →F
```

(E, , ,F,R) – found left end – state F
go right

```
--+---+---+---+---+---+---+---+---+--
  |   | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |
--+---+---+---+---+---+---+---+---+--
         ↑F
```

Halt – no quintuples with state F





Before and after shots of a run of the Turing Machine Copier program on a Turning Machine emulator

Finally, the Copier and Adder Turing Machines can be combined (with a suitable change of states etc.) so that after a number is copied, it's added to itself yielding a 3rd Turing Machine that doubles a number. Can multiplication be next?

It's not terribly difficult (?) to construct a Turing machine to subtract, multiply and/or divide two numbers,

Example Subtraction: 7 minus 3 starts might with the minuend (7) followed by the subtrahend (3).

```
---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+
...| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 |... |   |   |   |   |   |
---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+
    ↑A
```

The Turing Machine repeatedly cross off 1's from the minuend (7) and subtrahend (3)

```
---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+
...| 1 | 1 | 1 | 1 | x | x | x | 0 | x | x | x | 0 |… |   |   |   |   |   |
---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+
                                 ↑D
```

When the subtrahend is all crossed out, it *copies* the remaining minuend 1's to the end, resets the x's to 1's thus positioning the difference (4) after the minuend (7) and subtrahend (3) instead of overwriting them.

```
---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+
...| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 0 | 1 | 1 | 1 | 1 | 0 |   |
---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+---+
     ↑F
```

A machine to do division could be leveraged to determine if an integer was prime.

Essentially a Turing Machine is a computer and in fact a Turing machine can be programmed to do anything (very slowly) that any modern computer can do. In terms of the kinds of computation that can be performed, no modern computer is more powerful than a Turing machine (*Turing's Thesis*).

2. Turing's Thesis

In a more rigorous mathematical approach, a Turing Machine is completely defined by the sextuple $TM = (Q, \Sigma, \delta, q_0, \square, F)$ where …

Q is a finite set of internal states: $Q = \{q_1, q_2, q_3, ..., q_n\}$

Σ is a finite set of tape symbols: $\Sigma = \{a_1, a_2, a_3, ..., a_m\}$

δ is a transition function from $Q \times \Sigma \rightarrow Q \times \Sigma \times \{L, R\}$ (the quintuples)

$\square$ is the special blank symbol

$q_1 \in Q$ is the initial state

$F \subseteq \Sigma$ is a set of final states.

Turing's Thesis states that *any computation that can be carried out by mechanical means can be performed by some Turing Machine*. Though Turing's Thesis cannot be *proved* there is strong evidence supporting it.

1. Any program done on an existing digital computer can be done on a Turing machine.

2. No one has been able to suggest a problem solvable by an algorithm[2] that can't be solved on a Turing Machine.

3. No other model of *mechanical computation* has been shown to be more powerful than a Turing Machine.

Thus, in terms of *computation*, a Turing Machine is equivalent to a modern digital computer.

[2] An algorithm - an **ordered sequence** of **unambiguous and well-defined instructions** that **performs some task** and **halts in finite time**.

A Universal Turing Machine and Computable Numbers

Turing went one step further with his Turing Machine idea. He constructed a Universal Turing Machine (UTM), one which could execute another Turing Machine program encoded on its (the UTM's) tape. That is, given any Turing Machine *M*, an *encoded version of a M* along with an input string *w* for M could be written on the tape of the UTM with the UTM performing the execution of M on input string *w*. Obviously a very slow and complicated process as the UTM must scan the encoded quintuples of M for the correct *M quintuple* and apply it to *w* *within the UTM environment*.

If you think about it, this is how computer hardware executes a program plus data that resides in the memory of the computer.

The UTM required using a standard description for any and all Turning Machines. So if we assume without loss of generality that the set of all possible states Q for any Turing Machine can be written as

$$Q = \{q_1, q_2, q_3 ...., q_n\}$$

where q_1 is the initial state and q_2 is the final state and the set of all tape symbols Σ can be written as

$$\Sigma = \{a_1, a_2, a_3, ..., a_m\}$$

then we can encode q_1 as 1, q_2 as 11 and so on, encode the blank □ as 1, encode a_1 as 11, a_2 as 111 and so on L as a 1, R as a 11 and using 0 as a separator between quintuple elements, encode any quintuple as a binary integer.

Example: $(q_1, a_2, q_3, a_2, R) \leftrightarrow 1011101110111011$

Double 00's will be used as separators between quintuples.

In this manner any Turing Machine program can be encoded as a string of 0's and 1's – as a binary integer. (Note that not every binary string encodes a "legal" Turing Machine.) Hence there is an obvious one-to-one correspondence between a set of integers and the set of Turing Machines. Since the set of integers encoding Turing Machines is countably infinite, the set of Turing Machine is also countably infinite. If we define a computable number as one which can be computed by a Turing Machine, there are only countably many computable numbers.

However, since the set of real numbers form an uncountable set while the set of numbers computed by Turing Machines form a countable set, there must be numbers which cannot be computed. In a sense those un-computable numbers are unknowable.

The Entscheidungsproblem

Since any Turing Machine *M* with input *w* can be encoded as a program that can be run on the Universal Turing Machine UTM, Turing used this to solve in 1936 the *Entscheidungsproblem.* This problem asks the question

"Is there a mechanical method to decide a simple yes or no question within a certain domain of questions?"

For example, "Is an integer n prime?" its decidable! (If n cannot be divided by some integer between 2 and $\sqrt{n}$, then n in prime. See *Prime Syllable Song.*

The famous *Halting Problem* asks the question "Given any Turing Machine *M* and any input *w*, does there exist a method to determine if *M* starting on input *w* halts?" Turing proved that no such method exists – it's undecidable. Thus, the answer to the *Entscheidungsproblem* is no!

3. The Imitation Game - Turing Test – Artificial Intelligence - Can Machines Think?

In October 1950, in an article titled *Computing Machinery and Intelligence* Alan Turing introduced what he called the Imitation Game which was the basis for the Turing Test for Artificial Intelligence.

Turing's opening sentence reads "I PROPOSE to consider the question, 'Can machines think?'"

In the Imitation Game there is a man, a woman, and an interrogator (of either sex) are in separate rooms. It was the goals of the interrogator to determine which of the other two was the man and which was the woman by using a teletype to send questions and receive answers from either. The use of a teletype was to neutralize any outside differences (for example the sound of their voices) between the man and the woman. What made the game interesting was that the man's goal was to convince the interrogator that he was the woman while the woman of course was to answer truthfully that she was the woman. After a certain period of time the interrogator had to decide on the basis of the questions asked and the responses obtained, which of the two was the woman.

This was the basis for the Turing Test. However, instead of a man and a woman, one was a digital computer and the other a human both with the goal to convince the interrogator that they were the human.

And if the interrogator couldn't accurately determine which was which, who was to say that a computer can't think?

Or to put it more prosaically "If it looks like a duck, and acts like a duck and keeps company with ducks, it must be a duck"

"The test: if we can't distinguish that
They are not us, they must be us.
But do they give a shit about the sky?
They say they do."

4. The Turing Award

Named after Alan Turing, the Association of Computing Machinery (ACM), a (or the?) premier professional computer science association, annually bestows the Turing Award to "contributors of lasting and major technical importance to computer science". It's considered to be the "Nobel Prize in Computing". Winners are a veritable "Who's Who" of famous computer scientists.

Galatea, Helena, and Rachael ?

Galatea - Galatea began as an ivory statue created by Pygmalion of Cyprus who fell in love with is work. The statue was subsequently brought to life by Aphrodite and Pygmalion marries Galatea. Pygmalion is also the name of play by George Bernard Shaw and the basis for the musical *My Fair Lady.* There is an obvious similarity.

Helena – a female robot (robotess?) from the 1920 science-fiction *play* by the Czech writer Karel Čapek. "***R.U.R.***" stands for Rossumovi Univerzální Roboti (Rossum's Universal Robots). In the play the robots eventually overthrow and eliminate their human masters, and in the end become the new humanity.

The word "robot" was derived from the play R.U.R.

It is interesting that in 1920 Čapek foresaw one of the future nightmare scenarios for AI.

Rachael – an advanced female replicant (bioengineered humanoid) from the movie *Bladerunner* who eventually runs off with the protagonist Rick Deckard. In the movie replicants were engineered with short life spans – except for Rachael?

The two shortest Sci-Fi Stories every written

"Boy meets girl.
Boy loses girl.
Boy builds girl."

"Girl meets boy.
Girl loses boy.
Girl builds boy."

Measured Illusion

Euler - Mascheroni Constant
as n ⟶ ∞ then [1 + 1/2 + 1/3 + 1/4 + …+ 1/n] - ln(n) = γ

harmonic series, etch-a-sketch tracing along the logarithmic line
endless staircase cascading down a smoothed channel
cobble-stone road of stories worn into history
dubstep popping, swan-lake dance apace
steps and thread neatly separated
by one bare number, carried
119 billion places so far
we know it and don't
rational, irrational
transcendental
algebraic?
but we
know
it is
real

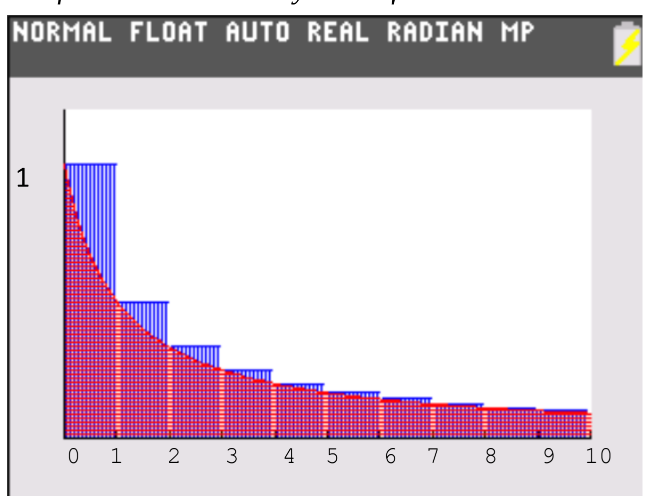
0.57721566490153286060651209008240243104215933593992…

-- E R Lutken (3: A Taos Press © 2021)

A Graphic Explanation - Rotate the poem 90 degrees counterclockwise.

The Euler-Mascheroni constant γ is the limit of the difference between the harmonic series and the natural logarithmic function ln(n+1) as *n* goes to infinity. *Separately* both the series and the function diverge to infinity as *n* gets large but *subtracting term by term one from the other term yields a finite value!*

The Euler-Mascheroni constant was originally defined as $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}-\ln(n+1)$ although today it's written in the form $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}-\ln(n)$ as given in the poem's subtitle. However, it's easier to visualize the *Euler-Mascheroni graphic* (on the right) using the original definition. In any case both definitions converge to the same value denoted by the Greek letter γ .



Graphically the *difference* between the harmonic series $\sum_{k=1}^{n}\frac{1}{k}$ and $\ln(n+1)$ is the blue area b*etween* the top of the "staircase cascading down" harmonic series (also seen in the layout of the poem) and the logarithmic function ("smooth channel"), the area in red covering the blue.

The **harmonic series** is the sum of the reciprocals of the integers; that is

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\ldots+\frac{1}{n}+\ldots.=\sum_{n=1}^{\infty}\frac{1}{n}$$

What is interesting about the harmonic series is that even though the *sequence of terms gets smaller and smaller*, their *sum grows without bound*, or to put it mathematically, *approaches infinity*. This unusual behavior has been known by mathematicians since the 13th century and is a standard counter example presented to students studying calculus to *contradict* the notion that if the terms of a sequence go to zero then their sum must converge to a finite value.

To show that the harmonic series grows without bound (i.e. diverges) we compare it to a similar but smaller series whose terms are the same (in red) or smaller (in black).

$$1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}+\frac{1}{9}+\frac{1}{10}+\frac{1}{11}+\frac{1}{12}+\frac{1}{13}+\frac{1}{14}+\frac{1}{15}+\frac{1}{16}+\frac{1}{17}+\ldots+\frac{1}{31}+\frac{1}{32}+\frac{1}{33}+\ldots$$

$$1+\frac{1}{2}+\frac{1}{4}+\frac{1}{4}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{32}+\ldots+\frac{1}{32}+\frac{1}{32}+\frac{1}{64}+\ldots$$

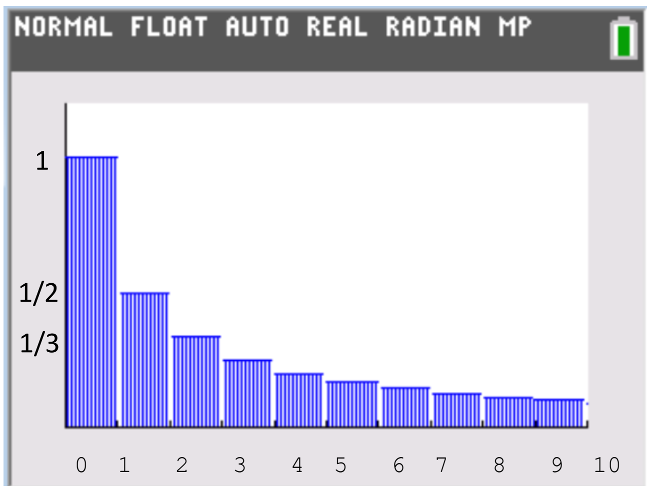
Notice that for the smaller series we can combine terms, in this case adding up the black terms before a red one (except for the leading 1) which always gives a value of $\frac{1}{2}$

$\frac{1}{2}$,

$\frac{1}{4}+\frac{1}{4}=\frac{1}{2}$,

$\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}=\frac{1}{2}$, and

$\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}+\frac{1}{16}=\frac{1}{2}$ etc.



The second smaller series is equivalent to $1+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots$ which obvious grows without bound (i.e. diverges to infinity). Thus, it follows that the larger harmonic series must also grow without bound (diverge to infinity). That is

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}=\infty$$

Graphically this can be seen as the (blue) areas under the step function where the heights of the steps are 1, $\frac{1}{2},\frac{1}{3}$, etc. the combined areas of which approach infinity as n increases.

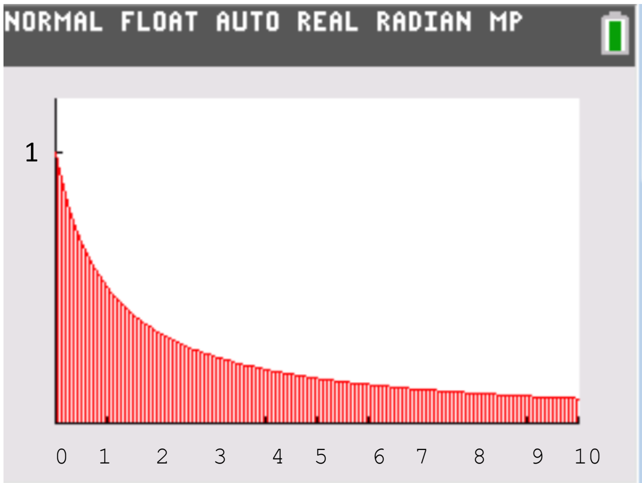
The **natural logarithmic function** denoted as $y=\ln(n)$ is the inverse of the exponential function $y=e^{x}$ (see *Phaethon's Ride*). From integral calculus it can also be shown that $\ln(n)$ is the area under the curve $y=\frac{1}{x}$ on the interval $[1,n]$, that is $\ln(n)=\int_1^n \frac{1}{x}dx$. Since $\frac{1}{x}$ on the interval $[n-1,n]$ is greater than $\frac{1}{n}$ we have the inequality

$$\ln(n)=\int_1^n \frac{1}{x}dx \geq \frac{1}{2}+\frac{1}{3}+...+\frac{1}{n}=\left(\sum_{k=1}^{n}\frac{1}{k}\right)-1.$$

Thus, there is close connection between $\ln(n)$ and the (partial) harmonic series. Since $\ln(n)$ is greater than the *partial harmonic series* minus 1, and since the hormonic series diverges to infinity as n increases, the natural logarithm also diverged to infinity as n increases.

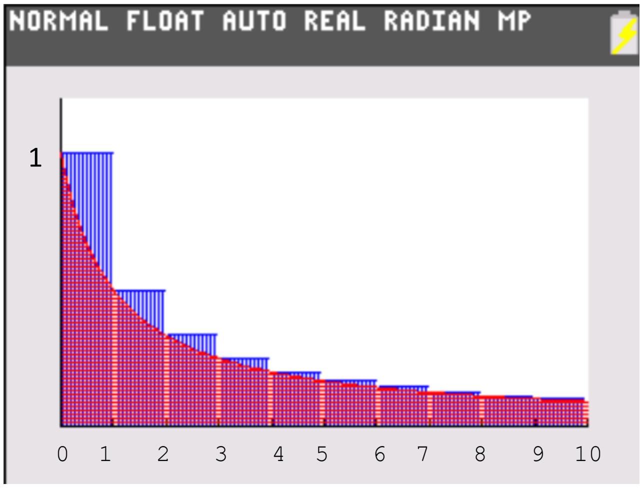
The function $\ln(n+1)$ is the function $\ln(n)$ shifted left by 1so it's defined by

$$\ln(n+1)=\int_1^{n+1}\frac{1}{x}dx=\int_0^n \frac{1}{x+1}dx$$



That is, it's the area under the curve $\frac{1}{x+1}$ on the interval $[0,n]$ so it grows without bound (diverges to infinity) as *n* get large. Thus $\lim_{n\to\infty}\ln(n+1)=\lim_{n\to\infty}\int_0^n \frac{1}{x+1}dx=\infty$.

Combining the blue graph of the Harmonic Series $\sum_{k=1}^{n}\frac{1}{k}$ by overlaying the red graph of $\ln(n+1)=\int_0^n \frac{1}{x+1}dx$ shows the area between the two: the Euler-Mascheroni function graphic.



Just like the terms of the Harmonic Series, the terms of the Euler-Mascheroni series gets smaller as n increases but does the Euler Mascheroni series diverge like the Harmonic Series?

The answer is no!

Showing $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}-\ln(n+1)$ converges to a finite value.

In mathematics, a theorem, the Monotone Convergence Theorem states that any bounded monotonically increasing or decreasing sequence converges to a unique finite value (either a *least upper bound* or a *greatest lower bound*) .

A sequence $s_1, s_2, s_3, ..., s_n, ...$ is *monotonically increasing* if and only if $s_n \le s_{n+1}$ for all n. (A similar definition holds for *monotonically decreasing*)

A sequence $s_1, s_2, s_3, ..., s_n, ...$ is *bounded above* if and only if for some number B, $s_n \le B$ for all n. (A similar definition holds for *bounded below*.)

The validity (truth) of the Monotone Convergence Theorem should be obvious since in the increasing case, the terms keep increasing but must *run up against* some bound (which may or may not be the initial bound B). In one sense it's an obvious (?) property of the *ordered real* numbers.

Let's define $s_n = \sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) = \sum_{k=1}^{n} \frac{1}{k} - \int_0^n \frac{1}{x+1} dx$ to be the n[th] partial sum for the Euler-Mascheroni series. We need to show two things:

1. $s_n \le s_{n+1}$ for all n is monotonically increasing and
2. $s_n \le B$ for all n for some bound B.

Both proofs which are somewhat technical are given below. It helps to note that $\frac{1}{n} > \int_n^{n+1} \frac{1}{x} dx > \frac{1}{n+1}$ That is the area under the curve $\frac{1}{x}$ on the interval $[n, n+1]$ (i.e. *between* n and $n+1$) is *sandwiched between* $\frac{1}{n}$ and $\frac{1}{n+1}$! We first show that s_n is a *monotonically increasing sequence* by showing the difference between consecutive terms $s_{n+1} - s_n$ is positive

1. $$s_{n+1} - s_n = \sum_{k=1}^{n+1} \frac{1}{k} - \int_0^{n+1} \frac{1}{x+1} dx - \left(\sum_{k=1}^{n} \frac{1}{k} - \int_0^n \frac{1}{x+1} dx \right) = \frac{1}{n+1} - \int_n^{n+1} \frac{1}{x+1} dx \ge 0$$

   since $\int_n^{n+1} \frac{1}{x+1} dx = \int_{n+1}^{n+2} \frac{1}{x} dx < \frac{1}{n+1}$. Thus $s_{n+1} - s_n \ge 0$ or $s_{n+1} \ge s_n$.

Therefore since $s_{n+1} \ge s_n$, the terms are monotonically increasing. We next show that the sequence s_n is bounded by 1.

2. $$\sum_{k=1}^{n} \frac{1}{k} = 1 + \sum_{k=2}^{n} \frac{1}{k} < 1 + \int_1^n \frac{1}{x} dx = 1 + \ln(n) < 1 + \ln(n+1)$$

   Therefore $\sum_{k=1}^{n} \frac{1}{k} - \ln(n+1) < 1$

So while separately the sum $\sum_{n=1}^{\infty}\frac{1}{n}=\infty$ and the integral $\int_0^{\infty}\frac{1}{x+1}dx=\infty$ are both infinite (it makes no sense to subtract one from the other as $\infty-\infty$ is undefined), the expression $\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}-\int_0^{n}\frac{1}{x+1}dx$ being in some sense the difference between these two expressions (which separately are infinite) is a finite number - the Euler-Mascheroni constant γ ! Mathematically, how odd!

"steps and thread neatly separated
by one bare number"

But what kind of number is Euler - Mascheroni Constant?

Numbers are either rational; that is they can be expressed as the quotient of two integers (for example $^7\!/_{17}$) or irrational, they can't be expressed as the quotient of two integers ($\sqrt{2}$, the length of the diagonal of a unit square being the historically famous example of an irrational number – see *Irrational Loss*).

We do not know if γ (gamma the Greek letter used to denote the Euler-Mascheroni constant) is rational or irrational (though I'd put my money on it being irrational).

But even if γ is irrational, we don't know if γ is *algebraic* or *transcendental*. Recall from *Meditation on Transcendental Numbers*, an *algebraic* number is the root of a polynomial with integer coefficients.

Transcendental numbers are mysterious, defined as being not algebraic; mysterious since we know in some hard mathematical sense that there are more transcendental numbers than algebraic numbers, but transcendental numbers are very hard to find. Hence

"we know it and don't
rational, irrational
transcendental
algebraic?
but we
know
it is
real"

The Completeness Axiom for Real Numbers

The set of real numbers is an *ordered set*; that is given any two real numbers *a* and b either $a=b$, $a<b$ or $a>b$.

The Monotone Convergence Theorem mentioned above was used to show that the Euler-Mascheroni constant $\gamma=\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}-\ln(n+1)<\infty$ was a finite number. Though it was argued that the Monotone Convergence Theorem was *obviously true* or at least was plausible, it does require a formal, mathematical proof. Such a proof requires some additional insight into the behavior of the *ordered real numbers* which is captured by an additional axiom, the Completeness Axion for Real Numbers; that is …

Completeness Axiom for Real Numbers: Every nonempty set S of real numbers which has an upper bound B has a least upper bound b; that is, if B is any upper bound for a non-empty set S of real numbers then there exists a unique least upper bound b such that for all numbers $n \in S$, $n \le b$ *and* $b \le B$.

Recall from the *Meditation on Transcendental Numbers* the nested Venn diagram of numbers: the set of Natural Numbers **N** contained in the set of Integers **Z** contained in the set of Rational Numbers **Q** contained in the set of Real Numbers **R** contained in the set of Complex Numbers **C** where finally we had algebraic closure for the operations of addition, subtraction, multiplication, division, exponentiation and n^{th} roots. It seemed we had somehow accounted for all numbers, but did we?

The *Completeness Axion for Real Numbers* "fills in or accounts for any holes" between numbers by making axiomatic that statement that if a set of numbers S is bounded above by some number B (which may or may not be in S), there exists a number *b* (which may or may not be in S) such that for every element *n* in the set B, $n \le b$ (the number *b* is called the *least upper bound*).

For example, consider the infinite set of numbers $S = \left\{ \frac{3}{4}, \frac{7}{8}, \frac{15}{16}, \frac{31}{32}, \ldots, \frac{2^n - 1}{2^n}, \ldots \right\}$ from the Zeno Paradox discussion in *Cantor's Ghazal* . Since this set is bounded by 1 (or any number greater than 1), the Completeness Axion for Real Numbers states that there is a least upper bound (which in this case is 1) to which the series converges. Note here that 1 is NOT an element of the set S. This is a mathematical justification behind the idea that *an infinite process can terminate in a finite value*.

The *Completeness Axion* is not the *Monotone Convergence Theorem*, indeed it's less specific (nothing about a monotone sequence of numbers). However, the Completeness Axiom is an axiom (i.e. a statement that is obviously true) which is used to prove the more specific Monotone Convergence Theorem which in turn guarantees that the Euler-Mascheroni constant exists.

The Rational Numbers are Incomplete.

By way of contrast, we can show that the set of rational numbers Q is *incomplete*; that is the *Completeness Axiom* does not hold when restricted to the rational numbers. There are bounded sets of rational numbers whose least upper bound is not rational but irrational.

For example, the irrational number $\sqrt{2} = 1.414213562....$ can be seen as the limit of a sequence of rational number approximations to $\sqrt{2}$. The sequence of *rational* numbers

$$\frac{14}{10}, \frac{141}{100}, \frac{1414}{1000}, \frac{14,142}{10,000}, \frac{141,421}{100,000}, \frac{1,414,213}{1,000,000}, \frac{14,142,135}{10,000,000}, \frac{141,421,356}{100,000,000}, \frac{1,414,213,562}{10,000,000,000} \ldots$$

converges to $\sqrt{2}$ but the least upper bound for this set is the *irrational* number $\sqrt{2}$. One way to think about the rational numbers is that as an *ordered set* it has a lot of holes that need to be plugged!

But the Monotone Convergence Theorem based on the Completeness Axiom for real numbers guarantees that there is *something* that the expression $\lim_{n \to \infty} \sum_{k=1}^{n} \left(\frac{1}{k} \right) - \ln(n+1)$ converges to although we're not sure if it's algebraic or transcendental - "*but we know it is real.*"

Φ – *E R Lutken* (3: A Taos Press © 2021)

Bewitching ratio, shadow of Fibonacci's

Line of jostled figures, ageless lovers in mad pursuit

Depleted of perfect squares at every turn

Forever denied the dull embrace of symmetry

In exchange for eternal mysterious fervor

The most irrational number: one plus one over

One plus one over one plus one over one plus one

Over one plus one over one plus one over one

On and on in a dizzying twilight-zone spiral

Reflections caught in vanity mirrors, proceeding

Toward anticipated static infinity

Within searching mortal eyes or minds, never quite there

Imperfect and incomprehensible perfection

Spiral arms of far galaxies

Iris of the hurricane's eye

Branching patterns of willow trees

Pineapples perfect checkered quilt

Petals of lilies, buttercups

Mesmerizing sunflower whorl

Twisting shells of ponderous snails

DNA's molecular turn

Enigmatic fact

Beautiful riddle

Secret Formula

The square root of five)

One half times (one plus

Grabbing light

Claiming space

Packing seeds

Spinning

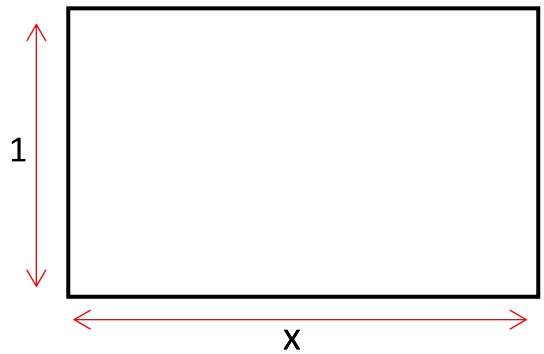
Towards

Still

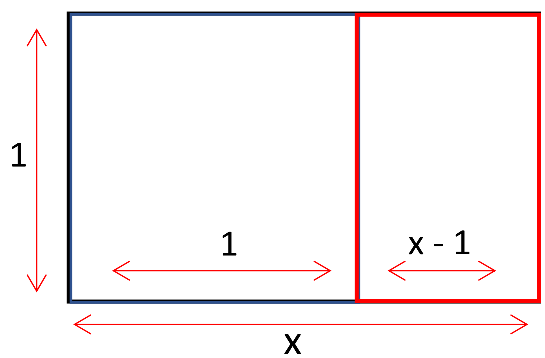
Peace

Mysterious ϕ

Consider the following problem. Start with an oblong rectangular object whose dimensions are x by 1 where x > 1,



Now divide the rectangle into a 1 x 1 square on the left and a 1 by (x-1) rectangle on the right.



If the smaller 1 by (x – 1) rectangle on the right has the same *proportion* as the larger x by 1 rectangle containing it, that is $x/1 = 1/(x-1)$ or $x^2 - x = 1$ what is the value of x?

Solving the quadratic equation $x^2 - x - 1 = 0$ using the quadratic formula gives you $x = \frac{1 \pm \sqrt{5}}{2}$. Since $\frac{1-\sqrt{5}}{2} \approx -0.6180339887$ is a negative value, the value of x is $\frac{1+\sqrt{5}}{2} \approx 1.618033989$ which we call ϕ, phi.

Secret Formula
The square root of five)
One half times (one plus

One half times (one plus The square root of five)

Moreover, the smaller rectangle on the right has the same *proportion* ϕ as the original. That is the long side 1 divided by short side $\phi - 1$ also equals ϕ.

$$\frac{1}{\phi-1}=\frac{1}{\frac{1+\sqrt{5}}{2}-1}=\frac{1}{\frac{-1+\sqrt{5}}{2}}=\frac{2}{-1+\sqrt{5}}=\frac{2}{-1+\sqrt{5}}\cdot\frac{-1-\sqrt{5}}{-1-\sqrt{5}}=\frac{-2-2\sqrt{5}}{1-5}=\frac{-2\left(1+\sqrt{5}\right)}{-4}=\frac{1+\sqrt{5}}{2}=\phi$$

The number φ has some very interesting algebraic properties:

First $1-\phi=1-\frac{1+\sqrt{5}}{2}=\frac{1-\sqrt{5}}{2}=\psi$ (psi) the other solution to the quadratic $x^2-x-1=0$.

Second: The reciprocal $\frac{1}{\phi}=\phi-1$

$$\frac{1}{\phi}=\frac{2}{1+\sqrt{5}}=\frac{2}{1+\sqrt{5}}\cdot\frac{1-\sqrt{5}}{1-\sqrt{5}}=\frac{2\left(1-\sqrt{5}\right)}{-4}=\frac{-1+\sqrt{5}}{2}=\frac{-1+\sqrt{5}}{2}+1-1=\frac{-1+\sqrt{5}}{2}+\frac{2}{2}-1=\frac{1+\sqrt{5}}{2}-1=\phi-1$$

Third: Two other equivalent expressions for ϕ.

$\phi=\frac{1}{\phi-1}$ which we've also shown using the smaller 1 by $\phi-1$ red rectangle above.

$\phi=1+\frac{1}{\phi}$ Starting with $\phi=\frac{1}{\phi-1}\Rightarrow\phi-1=\frac{1}{\phi}\Rightarrow\phi=1+\frac{1}{\phi}$

Fourth: Powers of ϕ result in a *Fibonacci-like sequence*.

If you square it $\phi^2=\left(\frac{1+\sqrt{5}}{2}\right)^2=\frac{1+2\sqrt{5}+5}{4}=\frac{6+2\sqrt{5}}{4}=\frac{3+\sqrt{5}}{2}=1+\frac{1+\sqrt{5}}{2}=1+\phi$

If you cube it $\phi^3=\phi\cdot\phi^2=\phi\cdot\left(1+\phi\right)=\phi+\phi^2=\phi\left(1+\phi\right)$.

A fourth power: $\phi^4=\phi\cdot\phi^3=\phi\left(\phi\left(1+\phi\right)\right)=\phi^2\left(1+\phi\right)$

In general, for integer n ≥ 2, $\phi^n=\phi^{n-2}+\phi^{n-1}=\phi^{n-2}\left(1+\phi\right)$. Note the Fibonacci-like values for the exponents

The Golden Rectangle

Given any rectangle whose length l is ϕ times its width w, that is $l=\phi\cdot w$ so that the ratio of length to width is $\frac{l}{w}=\frac{\phi\times w}{w}=\phi$, if you partition the rectangle into a $w\times w$ square the remaining $w\times\left(l-w\right)$ rectangle has the ratio length to width ratio $\frac{w}{l-w}=\phi$, the same ϕ ratio as the original rectangle.

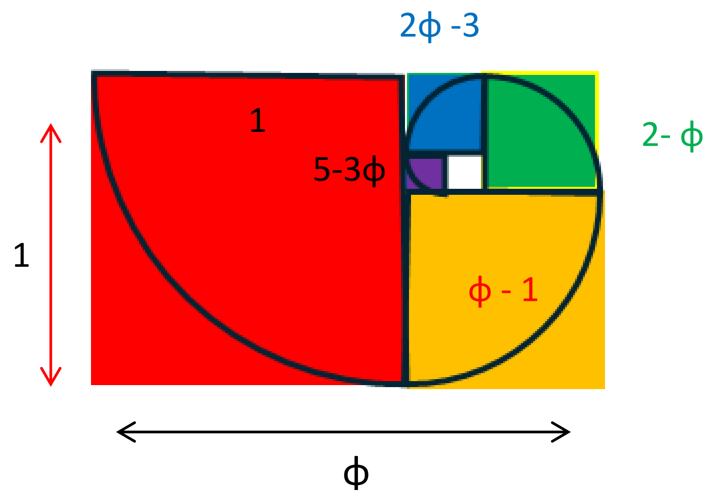
$$\frac{w}{l-w}=\frac{w}{\phi\times w-w}=\frac{w}{w\left(\phi-1\right)}=\frac{1}{\phi-1}=\phi$$

Repeat using the smaller length *w* by width *l-w* rectangle partitioning the smaller rectangle into a *(l-w)* × *(l-w)* square with a remaining length *(l-w) by width (2w-l)* rectangle which has the same length to width ratio ϕ. Thus the ratio of the smaller *(l-w) by (2w-l)* rectangle is

$$\frac{l-w}{2w-l}=\frac{\phi\times w-w}{2w-\phi\times w}=\frac{\phi-1}{2-\phi}=\frac{1}{(2-\phi)}\cdot\frac{1}{\phi}-\frac{1}{2\phi-\phi^2}=\frac{1}{2\phi-(1+\phi)}=\frac{1}{\phi-1}=\phi$$

using identities $\phi=\frac{1}{\phi-1}$ (or $\frac{\phi-1}{1}=\frac{1}{\phi}$) and $\phi^2=1+\phi$ from above.

The Golden Rectangle is the *scaffolding* for the poem Φ (seen below rotated with sides of length ϕ and 1). The embedded spiral is called the Fibonacci Spiral (quarter circles fitted to each square).



"Line of jostled figures, ageless lovers in mad pursuit
Depleted of perfect squares at every turn
Forever denied the dull embrace of symmetry
In Exchange for eternal mysterious fervor"

ϕ as a continued fractions

"The most irrational number: one plus one over
One plus one over one plus one over one plus one
Over one plus one over one plus one over one
On and on in a dizzying twilight-zone spiral"

$$\phi=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{\cdots}}}}}$$

There is a simple way to prove this result. Recall, from the initial $x/1 = 1/(x-1)$ equation we can obtain $x-1 = 1/x$ or $x = 1 + \frac{1}{x}$. From this we *recursively substitute* $1 + \frac{1}{x}$ for x on the right side. Thus

$$x = 1 + \frac{1}{x} \text{ to } x = 1 + \frac{1}{1+\frac{1}{x}} \text{ to } x = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{x}}} \text{ to } x = 1 + \frac{1}{1+\frac{1}{1+\frac{1}{1+\frac{1}{x}}}} \text{ etc.}$$

If you start with $x_0 = 1$ and recursively compute $x_{n+1} = 1 + \frac{1}{x_n}$ the resulting sequence will approach $\phi \approx 1.618033989$ (easily done on a handheld TI-84 graphing calculator initializing $1 \rightarrow \text{X}$ then repeating $1+1/\text{X} \rightarrow \text{X}$).

ϕ is constructable

A number is constructable if a line of that length can be constructed using only straight-edge and compass, a construction found in Euclid's *Elements* (ca 300 BCE).

$\phi = \frac{1+\sqrt{5}}{2}$ is constructable is follows.

Let $\overline{ab}$ is a line of length 1

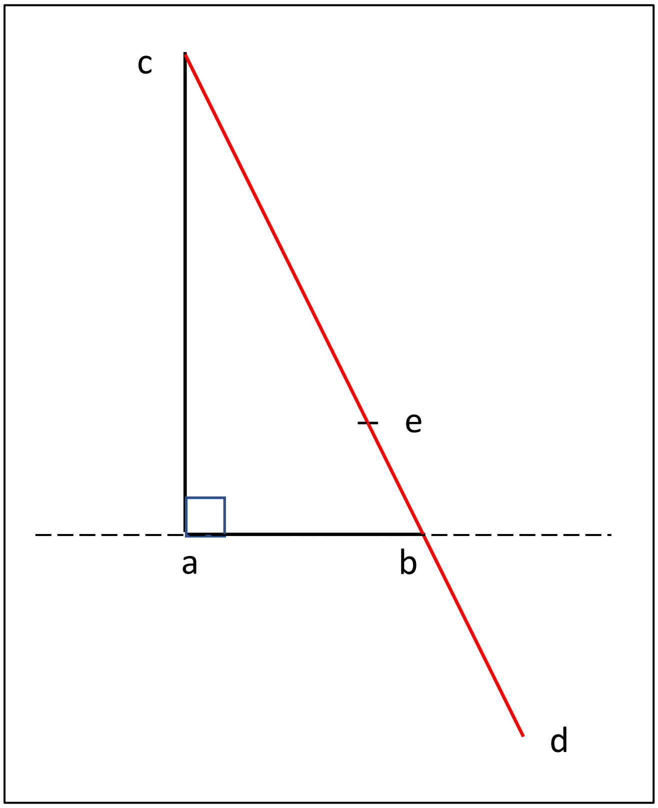
At point a construct a line perpendicular to $\overline{ab}$ extending the line so that $\overline{ac}$ has length 2.

Therefore Δabc is a right triangle and $\overline{bc}$ has length $\sqrt{5}$ by the Pythagorean Theorem.

Extend line $\overline{cb}$ to point d such that $\overline{bd} = 1$ so $\overline{cd}$ has length $1+\sqrt{5}$.

Bisect $\overline{cd}$ so that $\overline{ce} = \overline{ed} = \frac{1+\sqrt{5}}{2}$

All of these constructions used can be done using only straight-edge and compass.



Fibonacci Numbers

In the third section of his book *Liber Abaci* written in 1202, Leonardo of Pisa (a.k.a. Leonardo Bonacci, a.k.a. Leonardo Bigollo Pisano) but better known as Fibonacci (1170 – 1250) posed the following problem.

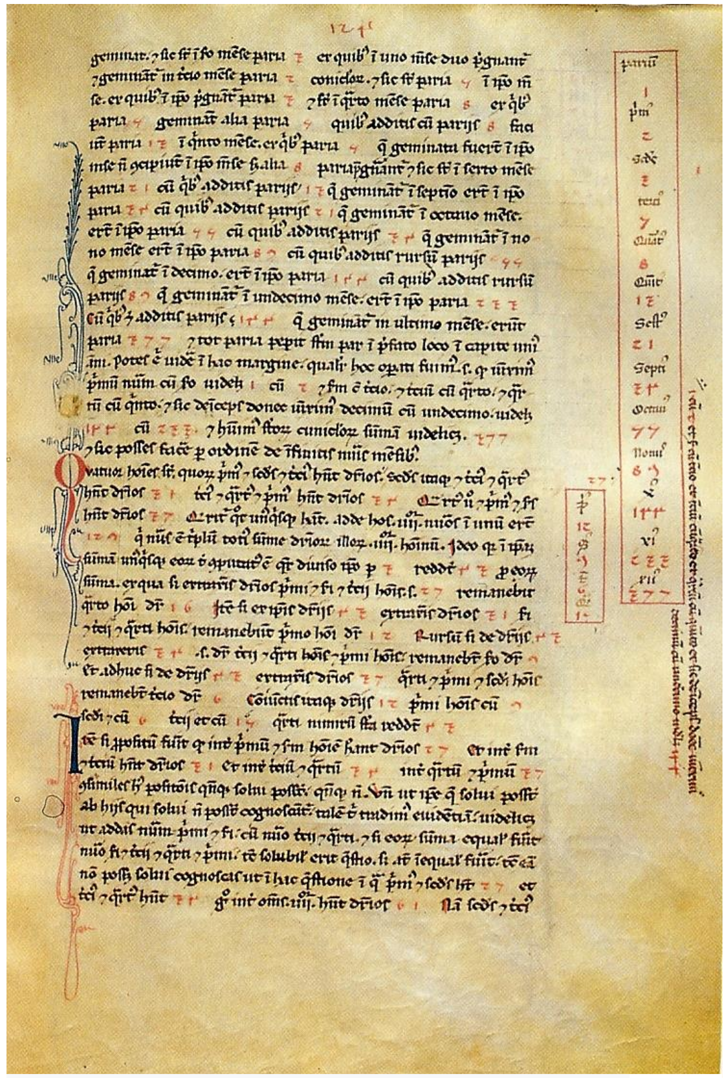
"A certain man put a pair of rabbits in a place surrounded on all sides by a wall. How many pairs of rabbits can be produced from that pair in a year if it is supposed that every month each pair begets a new pair which from the second month on becomes productive?"
https://mathshistory.st-andrews.ac.uk/Biographies/Fibonacci/

At month one there is a pair of immature rabbits. At month two they mature and breed to produce another pair of immature rabbits so that at month three we have 2 pair of rabbits.

At month four we have 3 pair – 2 mature and 1 immature pair (produced by the previous month's mature pair).

By month five the two previous mature pairs have produced two more immature pairs and the immature pair matures for a total of 5 pairs – 3 mature (one newly) and 2 immature pairs.

By month six we have 8 pairs of rabbits, 5 now mature and 3 produced by the previous 3 mature pairs.



*A page of Fibonacci's Liber Abaci from the Biblioteca Nazionale di Firenze showing (in box on right) the Fibonacci sequence with the position in the sequence labeled in Latin and Roman numerals and the value in Hindu-Arabic numerals.- This work is in the **public domain** in its country of origin*

A little thought suggests that the number of rabbits for the current month is the sum of the pairs from the previous month plus the new immature pairs from the mature pairs two month previously.

So, if F_n is the number of rabbit pairs at month n, then $F_1 = 1,\ F_2 = 1,\ and\ F_n = F_{n-1} + F_{n-2}\ \ for\ n \geq 2$

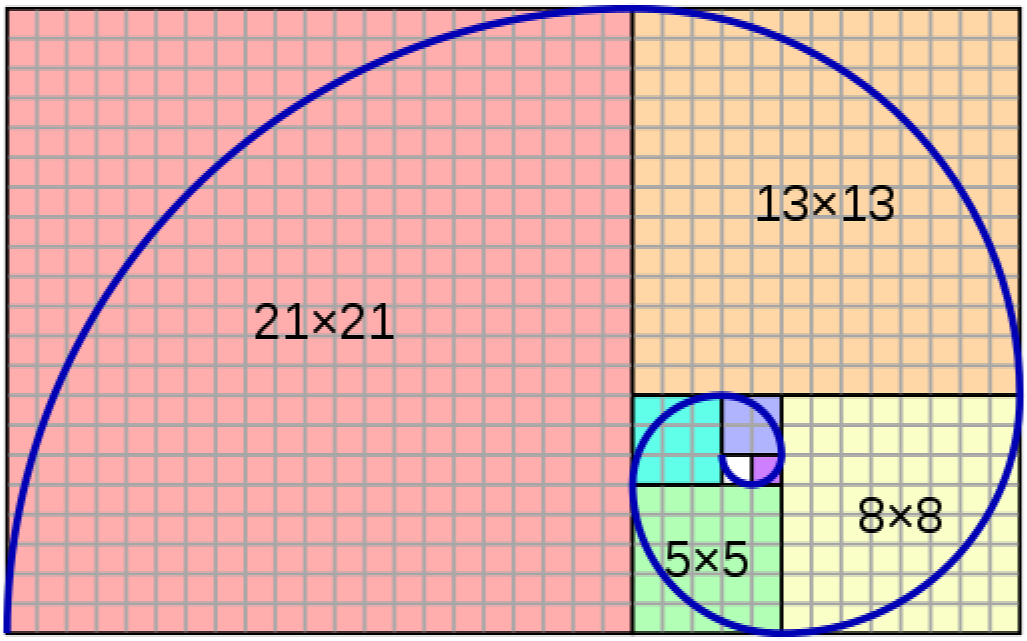
This generates the Fibonacci sequence: 1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, … which is closely connect with ϕ (recall the powers of ϕ equation).

"Bewitching ratio, shadow of Fibonacci's
Line of jostled figures… "

ϕ, ψ, and Fibonacci Numbers

Binet's formula: The Fibonacci recurrence expression $\begin{array}{l} F_0 = 0 \\ F_1 = 1 \\ F_n = F_{n-1} + F_{n-2} \quad n \geq 2 \end{array}$ can be express as the closed form expression $F_n = \dfrac{\phi^n - \psi^n}{\sqrt{5}} = \dfrac{\dfrac{\left(1+\sqrt{5}\right)^n - \left(1-\sqrt{5}\right)^n}{2^n}}{\sqrt{5}}$ for computing the n^th^ Fibonacci number.

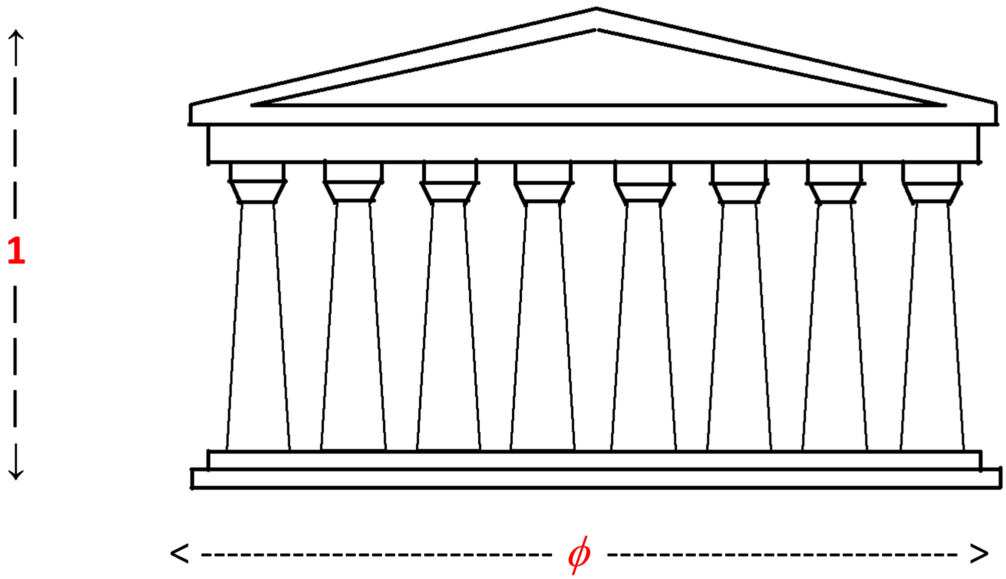
A Fibonacci Spiral



File:FibonacciSpiral.svg - Wikimedia Commons

The Golden Ratio ϕ in Art and Nature

Much has been made about the appearances of the Golden Ratio (or Golden Rectangle) in art and nature. For example, the length and height of the Greek Parthenon are approximately those of ϕ to 1.



Fibonacci spirals seem to occur in storms,  galaxies, 

shells,  and flowers. 

“Spiral arms of far galaxies
Iris of the hurricane’s eye
Branching patterns of willow trees
Pineapples perfect checkered quilt
Petals of lilies, buttercups
Mesmerizing sunflower whorl
Twisting shells oof ponderous snails
DNA’s molecular turn”

I guess the questions is – is there really something to this?

Phaethon's Ride

(e = 1/0!+1/1!+1/2!+1/3!+1/4!+1/5!+…)

Secret heart of death and growth,
exponential path of youth
mindless of the track of day.
No green fortitude can stay chargers freed from temperate yokes.

Lost in Helios' uncoiled oath,
Phaethon, pulled through fiery froth,
strains to tame the disarray,
the secret heart.

Wild upheaval, steep'ning slope
inverts towards the valley's throat.
Burning chariot falls away,
scorching trails of ash, decay,
leaving, amid seas of smoke,
the secret heart. -- E R Lutken (3: A Taos Press © 2021)

Who was Phaethon?

To understand the connection of Phaethon with the constant e, we need first to review the Greek story of Phaethon's Ride.

Phaethon was the son of the Greek god Helios who drove the chariot of the sun. When Phaethon approached Helios asking for assurance that he was indeed Helios's son, Helios swore an oath to grant Phaethon whatever he wanted. Phaethon asked to drive the chariot of the sun for one day which Helios reluctantly granted. Not surprisingly Phaethon was unable to control the chariot which first resulted in *climbing too high thus freezing the earth* then *diving too low thus burning the earth*. To save the earth from further damage Zeus struck Phaeton down with a thunderbolt.

> "Lost in Helios' uncoiled oath,
> Phaethon, pulled through fiery froth,
> strains to tame the disarray,
> the secret heart."

The number e

The number e, like the number π, arises naturally in mathematics although its origin is not as easily seen. Its value can be expressed as the sum of the infinite series …

$$e=\sum_{k=0}^{\infty}\frac{1}{k!}=\frac{1}{0!}+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+...+\frac{1}{k!}+...=1+1+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+...\approx 2.71828182...$$

e and compound interest

Suppose you invested x dollars at r % interest per year. After one year you would have the original amount invested *x* plus the interest accrued $r \cdot x$ or $x + r \cdot x = x(1+r)$ dollars. Reinvest this new amount to obtain after two years $x(1+r) + x(1+r)r = x(1+r)^2$. And in general, after *t* years, you would have $x(1+r)^t$ dollars – thus the power of compound interest.

Using PV (present value) to be the original amount invested, the future value, FV, after *t* years is given by the equation.

$$FV = PV(1+r)^t$$

Now let's modify the problem somewhat.

Instead of compounding annually, let's compound k times per year. This changes the formula to

$$FV = PV\left(1+\frac{r}{k}\right)^{k \cdot t}$$

Note that we divide the interest rate by *k* but increase the *compounding* by a factor of *k*.

Now consider the case where we start out with PV = \$1.00 and the rate *r* is 100% interest. It's intuitively obvious that after one year (k = 1) that the future value FV would be \$2. However, if we compound quarterly, we get interest after three months which is included in computing the interest for the next three months and so on … so that at the end of the year $\left(1+\frac{1}{4}\right)^4 = \$2.44$. If we compound monthly, we would get $\left(1+\frac{1}{12}\right)^{12} = \$2.61$, compounding daily using 365, $\left(1+\frac{1}{365}\right)^{365} = \$2.71$ and so on. Now as the number of compounding periods, *k*, increases without bound (*compounding continuously?*) we approach a limit: $\lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k = 2.718281828 = e$.

Exponential Growth and Decay

"Secret heart of death and growth,
exponential path of youth
mindless of the track of day.
No green fortitude can stay chargers freed from temperate yokes."

Equations of the form $y = a \cdot b^x$ where $a \neq 0$, $0 < b \text{ and } b \neq 1$ are the equations of *exponential* growth (for b > 1) and *exponential* decay (0 < b < 1). *Exponential* because the independent variable *x* is the exponent. The positive constant *b* is called the *base*.

Exponential growth equations govern phenomena like compound interest (e.g. $FV = PV(1+r)^t$) and population growth. Exponential decay equations govern phenomena like radioactive decay often expressed in terms of the half-life of a radioactive element. For example, the remaining amount (quantity) Q of a radioactive element is given by the equation $Q = Q_0 \left(\frac{1}{2}\right)^{t/h}$ where Q_0 is the original amount and h is the half-life.

The origins of e^x

In calculus, the exponential function $y = e^x$ where *e* is the *base* has some very unique properties, the foremost of which is that it is its own derivative; that is $\frac{d}{dx}e^x = e^x$ In fact all of its derivatives are the same making $y = e^x$ a very unique function. Because this is so, its Taylor Series expansion, obtained from its derivatives yields the infinite series expansion …

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \ldots + \frac{x^k}{k!} + \ldots$$

And of course, setting x = 1 yields the value for e as the limit of an infinite series

$$e = \sum_{k=0}^{\infty} \frac{1}{k!} = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots + \frac{1}{k!} + \ldots = 1 + 1 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \ldots \approx 2.71828182\ldots$$

To put this in a wider context, while π is the ratio of the circumference of a circle to its diameter, e is the base of the exponential function $y = e^x$ all of whose derivatives are the same. As it turns out, this is mathematically very useful!

Exponential Growth

"Wild upheaval, steep'ning slope"

Exponential growth functions climb very fast outracing, outpacing other common-place functions like polynomials. Indeed, the larger the x the faster $y = e^x$ increases – faster and faster.

Exponential Decay

"inverts towards the valley's throat."

But exponential growth can be turned around if we allow e^x to be in the denominator; that is as an *exponential decay* function like $y = e^{-x} = \frac{1}{e^x}$. Just as e^x increases faster and faster as x increases, the exponential decay function $e^{-x} = \frac{1}{e^x}$ decreases faster and faster as x increases, but (and this is important), it never equals zero and never crosses the x-axis (never hitting the ground unless struck by a lightning bolt).

"Burning chariot falls away,
scorching trails of ash, decay,
leaving, amid seas of smoke,
the secret heart."

$y = e^x$ *and its inverse* $y = \ln(x)$

As mentioned in *Measured Illusion*, the exponential function $y = e^x$ and the natural logarithmic function $y = \ln(x)$ are inverse functions. Since $y = e^x$ is a one-to-one function (i.e. each x value is mapped to a unique y value) it has an inverse $y = \ln(x)$.

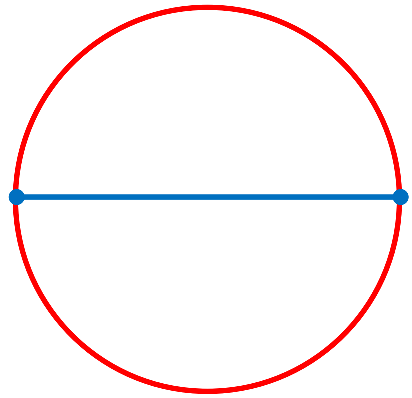
Formally we define $y = \ln(x)$ if and only if $x = e^y$ (note the reversal of roles for x and y). Since they are inverse functions, the one undoes the other: $x = e^{\ln(x)}$ and $x = \ln(e^x)$

The derivative of $y = \ln(x)$ is interesting: $\frac{d}{dx}\ln(x) = \frac{1}{x}$. Therefore, the integral $\int_1^n \frac{1}{x}dx = \ln(n)$ since integration is the *inverse* of differentiation. Recall in *Measured Illusion*, we introduced the logarithm ln(n) as the area under the curve $y = 1/x$ on the interval between 1 and n.

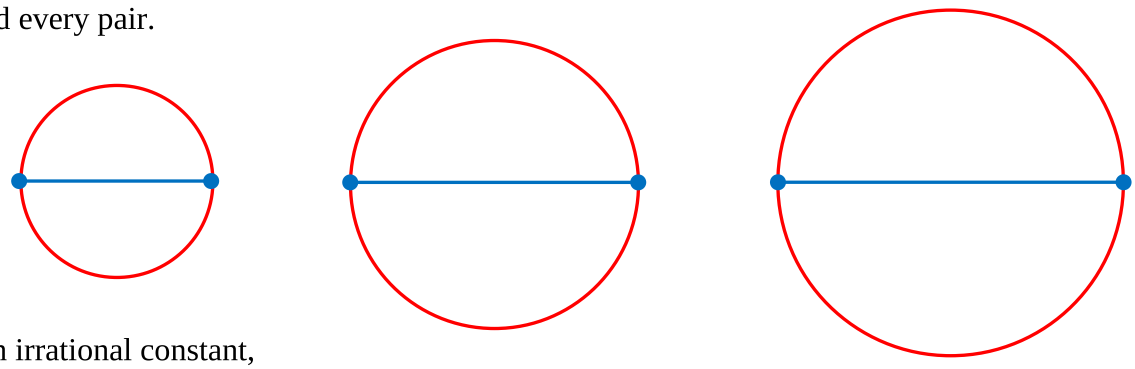
And while the exponential growth function $y = e^x$ increases faster than any power of x (e.g. $y = x^n$ for n > 1) as x increases, its inverse function $y = \ln(x)$ increases very slowly exhibiting what is called *logarithmic growth*. For example, $y = \ln(x)$ increases slower than the linear function $y = mx + b$ root functions like $y = \sqrt{x}$ or $y = \sqrt[n]{x}$ for n = 2,3,4,…

π

Trapped on planes,
circles and lines find each other,
centered, locking arms.



The faithful connection
always, always the same
for each and every pair.



Ruled by an irrational constant,
its measure, a sequence
never repeating. -- E R Lutken (3: A Taos Press © 2021)

$$\pi = \frac{\textit{circumference}}{\textit{diameter}} \approx$$

3. 14159 26535 89793 23846 26433 83279 50288 41971 69399 37510
58209 74944 59230 78164 06286 20899 86280 34825 34211 70679
82148 08651 32823 06647 09384 46095 50582 23172 53594 08128
48111 74502 84102 70193 85211 05559 64462 29489 54930 38196
44288 10975 66593 34461 28475 64823 37867 83165 27120 19091
45648 56692 34603 48610 45432 66482 13393 60726 02491 41273
72458 70066 06315 58817 48815 20920 96282 92540 91715 36436
78925 90360 01133 05305 48820 46652 13841 46951 94151 16094
33057 27036 57595 91953 09218 61173 81932 61179 31051 18548
07446 23799 62749 56735 18857 52724 89122 79381 83011 94912
98336 73362 44065 66430 86021 39494 63952 24737 19070 21798
60943 70277 05392 17176 29317 67523 84674 81846 76694 05132
00056 81271 45263 56082 77857 71342 75778 96091 73637 17872
14684 40901 22495 34301 46549 58537 10507 92279 68925 89235
42019 95611 21290 21960 86403 44181 59813 62977 47713 09960
51870 72113 49999 99837 29780 49951 05973 17328 16096 31859
50244 59455 34690 83026 42522 30825 33446 85035 26193 11881
71010 00313 78387 52886 58753 32083 81420 61717 76691 47303
59825 34904 28755 46873 11595 62863 88235 37875 93751 95778
18577 80532 17122 68066 13001 92787 66111 95909 21642 01989

A short history of π

The origins of π lie in early attempts to construct a square with the same area as a given circle, referred to as *squaring the circle*. This was one of the three famous classical problems of Greek mathematics, the other two being *trisecting an angle* and *doubling the volume of a cube*. But the problem goes way back before the Greeks of the 4th century BCE.

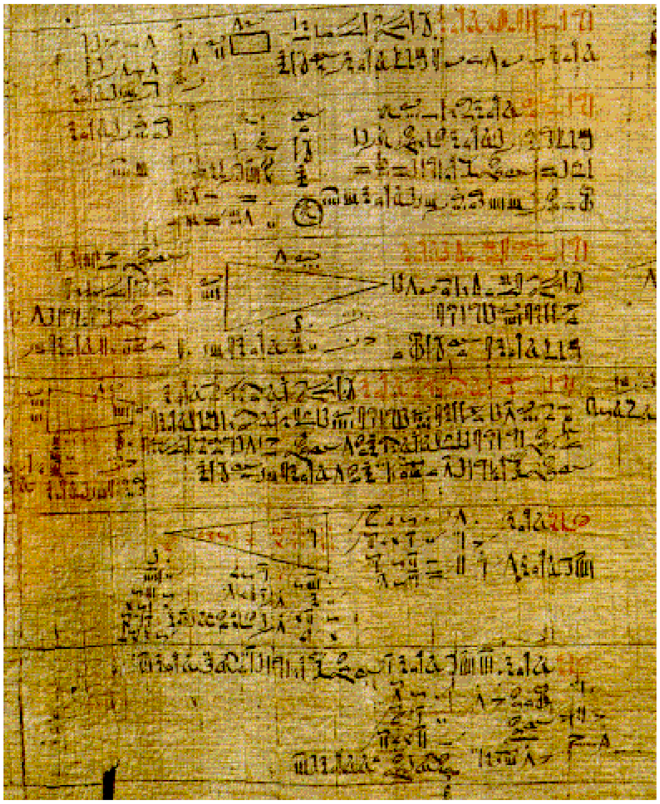
In his book *Journey Through Genius – Great Theorems of Mathematics,* William Dunham outlines four phases in the determination of π.

1 - Early Approximations

Well before the 4th century BCE Greeks, several ancient cultures came up with a number of pre-scientific approximations to π, probably caused by attempts (?) to determine the area of a circle by constructing a square with the same area (*squaring the circle*). Proposition XII.2 of Euclid's Elements (ca. 300 BCE) states "Circles are to one another as the squares on their diameters" so even if the ratio of the circumference to the diameter of a circle (i.e. the definition of π) was not known, they understood this particular relationship between the diameter of a circle and its area.

The ancient Babylonians used (?) the value of 3 for the ratio of the circumference to the diameter for a circle though there is some evidence that they might have used $3\frac{1}{8} \approx 3.1253$ instead. Writing about the "sea", a large circular basin for holding water, I Kings 7:23 from the Bible states "Then he made the molten sea, ten cubits from brim to brim, while a line of 30 cubits measured it around" which yields a value for 3.0 for π. There was some thought that this value originated with the Babylonians.

A more accurate approximation can be obtained from Problem 50 from the Egyptian Rhine papyrus (c.a. 1650 BCE – on the right) which states that a round field of diameter 9 *khet* (radius 9/2) has an area of 64 *setat* (square khet?). Given that $Area_{circle} = \pi r^2$ using $radius = \frac{9}{2}$ and a little algebra shows that



$$\pi\left(\frac{9}{2}\right)^2 = 64$$

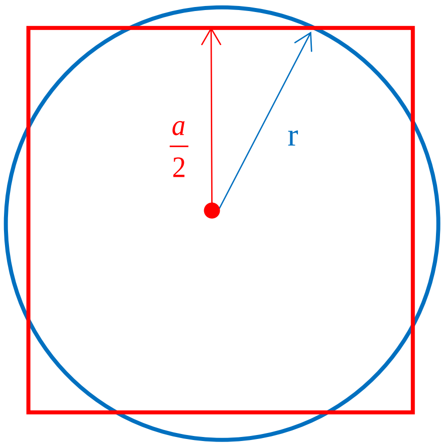
and so

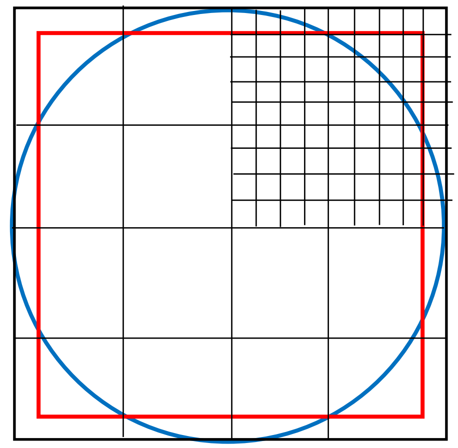
$$\pi \simeq \left(\frac{16}{9}\right)^2 = \frac{265}{81} = 3.1604938$$

So using the diameter of a circle

$$area_{circle} = \left(\frac{8}{9} \times diameter\right)^2$$

An interesting conjecture suggests a possible way of understanding where this calculation came from. From the figure on the left, the area of the blue circle and the area of the red square look very much the same. So how best to adjust the radius of the circle r and the width of the square a so that two areas are equal?





A simple way is to expand the red square by a factor of $\frac{9}{8}$ as was done on the right (black square) to match the diameter of the circle. The 9 by 9 grid overlay in the upper right quadrant seems to confirm that this ratio "works"; the area of a circle is *approximately* the same as the area of a square whose side is $\frac{8}{9}$ times the diameter of the circle. Thus

$$area_{circle} = \left(\frac{8}{9} \times diameter\right)^2$$

See: Engles, H; Quadrature of the Circle in Ancient Egypt; Historia Mathematica; Vol 4;1977;137-140

2 - Archimedean Methods

Archimedes (287-212 BCE) made two very important contributions to the history of the determination of π. First, he proved that area of a circle is one half the radius times the circumference; that is …

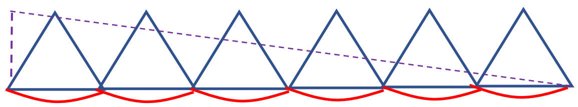
$$Area_{circle} = \frac{1}{2} r \times C$$

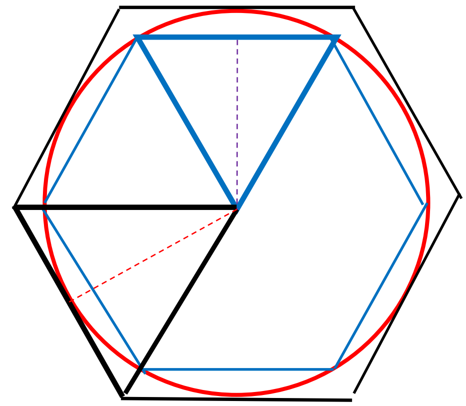
Since $C = 2\pi r$ this agrees with the modern formula for the area of a circle: $Area_{circle} = \pi r^2$.

Second, he computed the ratio of the circumference to the diameter, that is π, to be greater than $3\frac{10}{71}$ and less than $3\frac{1}{7}$. That is

$$3.140845 \approx 3\frac{10}{71} < \pi < 3\frac{1}{7} \approx 3.142857$$

To obtain the area, a circle can be *bracketed* by areas of inscribed and circumscribed regular polygons. If we *unroll* the triangles, we see that the area of a polygon is one-half the altitude of a triangle times the perimeter.





As the number of sides increase, the two polygons squeeze down on the circle. In the limit the two polygon areas converge to equal the area of the circle. It was a very clever proof – an example of what I call "proto-integration". However, the use of calculus by which this is easily done today (and which in some sense Archimedes' method prefigures) lay 2000 years in Archimedes' future.

The same diagram was also used to obtain the ratio of the circumference to the diameter of a circle. In the 3rd proposition from *Measurement of a Circle* Archimedes states "The ratio of the circumference of any circle to its diameter" (what we call π) "is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$ "
Starting with a circle of radius one (and diameter two), an inscribed hexagon has perimeter six which is obviously less than the circumference of the circle so $\pi > 3$. A simple bit of geometry can show that if s is the length of the side for an n-sided polygon with radius one, then $t = \sqrt{2-\sqrt{4-s^2}}$ is the length of the side of a 2n-sided polygon. So, for a hexagon with side $s = 1$, t = 0.5176380902 for a 12-sided polygon resulting in $\pi > 3.10582541$. Archimedes repeated this calculation for 24, 48 and finally a 96 sided polygon to get a lower bound of $3.140845 \approx 3\frac{10}{17}$.

A similar approach starting with a circumscribed hexagon and doubling the sides through 12, 24, 48 and 96 (requiring a different formula for finding the length of one side of a circumscribed 2n-gon given the length of a side of a circumscribed n-gon) yielded an upper bound of $3.142857 \approx 3\frac{1}{7}$ Thus

$$3.1408454 \approx 3\frac{10}{71} < Area_{circle} < 3\frac{1}{7} \approx 3.142857$$

Thus, we have a mathematically rigorous method for calculating π – subject only to the difficulty of the calculational effort required. Indeed, the record for determining the digits of π using this method goes to Ludolph Van Ceulen (1540 – 1610) who computed π to 35 digits.

3 – Arctangent Series and the Calculus

By the 16th and 17th centuries advances in mathematics yielded a number of equations for determining π . Probably the most famous is the Gregory (1638-1675)-Leibniz (1646-1716)–Nilakantha (c.a. 1450 – c.a. 1550) formula

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} ... = \sum_{k=0}^{\infty} \frac{(-1)^k}{2k+1}$$

A *derivation* of this formula is a very straightforward evaluation of the arctangent integral using the fact that $\frac{1}{1+t^2}$ can be rewritten as an alternating series which can be easily integrated term by term.

$$\arctan(x)=\int_0^x \frac{1}{1+t^2}dt=\int_0^x 1-t^2+t^4-t^6+...dt=x-\frac{x^3}{3}+\frac{x^5}{5}-\frac{x^7}{7}+...=\sum_{k=0}^{\infty}\frac{(-1)^k}{2k+1}x^{2k+1}$$

Note that arctan(x) is an alternating series of x raised to odd powers. Since $\arctan(1)=\frac{\pi}{4}$, evaluating the series for x = 1 give you $\frac{\pi}{4}$. Unfortunately, the Gregory-Leibniz- Nilakantha series for $\frac{\pi}{4}$ when x =1 converges very slowly.

However, John Machin’s 1706 arctangent-based formula seen below uses smaller values for x.

$$\frac{\pi}{4}=4\arctan\left(\frac{1}{5}\right)-\arctan\left(\frac{1}{239}\right)$$

With the smaller values of x ($\frac{1}{5}$ and $\frac{1}{239}$), the two infinite series used for the arctangents *converge faster*. For example, for $x=\frac{1}{5}$

$$\arctan\left(\frac{1}{5}\right)=\int_0^{1/5}\frac{1}{1+t^2}dt=\int_0^{1/5}1-t^2+t^4-t^6+...dt \text{ an}$$

$$=\frac{1}{5}-\frac{\left(\frac{1}{5}\right)^3}{3}+\frac{\left(\frac{1}{5}\right)^5}{5}-\frac{\left(\frac{1}{5}\right)^7}{7}+...$$

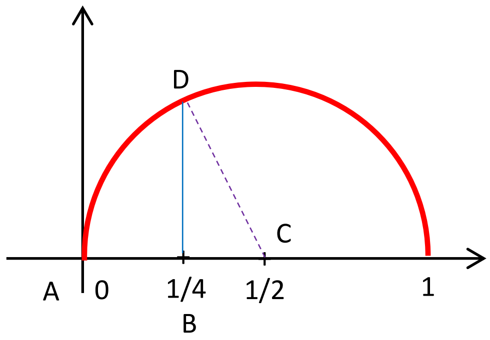
$$=\frac{1}{5}-\frac{1}{3\cdot 5^3}+\frac{1}{5\cdot 5^5}-\frac{1}{7\cdot 5^7}+...=\sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)\cdot 5^{2k+1}}$$

The increasingly large, odd powers of 5 in the denominator of each term causes the series for $\arctan\left(\frac{1}{5}\right)$ to converge faster compared to the series for $\arctan(1)$. This is also true (more so) for the terms of the series for $\arctan\left(\frac{1}{239}\right)$. The genius behind Machin’s 1706 equation (and others to follow) was coming up with an alternate expression for $\frac{\pi}{4}$ using faster converging arctangent series.

Newton’s Approximation to π (written 1671? – pub. 1737)

Using the newly discovered (invented?) calculus and his generalized Binomial Theorem, Newton easily (?) determined π to 16 decimals (3.1415926535897928) as follows.

Starting with the curve $y=\sqrt{x-x^2}=x^{1/2}\cdot(1-x)^{1/2}$ which is the equation of a half circle of radius $\frac{1}{2}$ centered at $\left(0,\frac{1}{2}\right)$, Newton was able to compute the area of the sector ADC which being $\frac{1}{6}th$ of the area of a circle of radius $\frac{1}{2}$ equaled $\frac{1}{6}\times\pi\left(\frac{1}{2}\right)^2=\frac{\pi}{24}$. This sector whose area was $\frac{\pi}{24}$ could be partitioned into the right triangle BCD and the area under the circular curve on the interval $\left[0,\frac{1}{4}\right]$



Since angle BCD is $60°$, BD is length $\frac{\sqrt{3}}{2}$ making the area of the triangle BCD $\frac{\sqrt{3}}{4}$. Using calculus, the area under the curve AD from 0 to ¼ is given by the integral $\int_0^{1/4} x^{1/2}(1-x)^{1/2}\,dx$.

$$\text{Thus } \frac{\pi}{24}=\frac{\sqrt{3}}{4}+\int_0^{1/4} x^{1/2}(1-x)^{1/2}\,dx$$

Now, using his Generalized Binomial Theorem, the integrand can be rewritten as an infinite series …

$$\begin{aligned}
y &= x^{1/2}(1-x)^{1/2}\\
&= x^{1/2}\left(1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^4-\frac{7}{256}x^5-\ldots\right)\\
&= x^{1/2}-\frac{1}{2}x^{3/2}-\frac{1}{8}x^{5/2}-\frac{1}{16}x^{7/2}-\frac{5}{128}x^{9/2}-\frac{7}{256}x^{11/2}-
\end{aligned}$$

which can be expanded to any degree of accuracy and integrated term by term. This value plus the value of the triangle $\frac{\sqrt{3}}{4}$ equals $\frac{\pi}{24}$ which can be solved for π.

Newton wrote "*I am ashamed to tell you to how many figures I carried these calculations, having no other business at the time*".

Further hand calculations of π

The origin of the symbol for pi, π, first appeared in a 1706 work by Willian Jones (1675 – 1749). The use of the symbol π was further popularized by Leonard Euler (1707 – 1783).

John Machin in 1706 using his arctangent-based formula $\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$ determined π to 100 digits.

In 1767 Johann Heinrich Lambert (1728 – 1777) proved that π is irrational; that is π could not expressed as the quotient of two integers.

William Shanks in 1853 determined π to 605 digits then in 1874 determined π to 707 digits.

In 1882 Ferdinand Lindemann (1852 – 1939) proved that π was transcendental! As mentioned in *Meditation on Transcendental Numbers*, that π was proved to be transcendental resulted in showing that the *circle could not be squared*.

In *A Budget of Paradoxes Vol. II* (published posthumously in 1873) Augustus De Morgan (1806 – 1871) observed that the number of 7's in Shanks's 1853 determination of π did not occur as often as the other digits. It was later discovered that there were errors in Shanks's determination beginning at the 527th digit which D. F. Ferguson in corrected 1947 determining π to 710 digits thus correcting Shanks's error.

In September 1947 D. F. Ferguson and John Wrench determined π to 808 digits using a desk calculator and a different arctangent formula

$$\frac{\pi}{4} = 8\arctan\frac{1}{10} - \arctan\frac{1}{239} - 4\arctan\frac{1}{515}$$

In 1949 Levi Smith and John Wrench determined π to 1,120 digits using desk calculators.

Augustus DeMorgan's observation of the relative scarcity of 7's in Shanks's erroneous 1853 determination of π led to the question as to whether the digits in π occur equally often; that is, in the long run do the number of 0's, 1's 2's etc. occur, in an asymptotic sense, equally often? Thus, is π normal – meaning are the digits *uniformly distributed*? The answer is not known and there seems to be no way of proving or disproving this assertion although the counts of digits obtained from larger and larger determinations of π seem to support the assertion that π is normal.

4 - Enter the Computer - The 1949 ENIAC Determination of π

In 1949, at the suggestion of John von Neumann, a team headed by George Reitwiesner at the U.S. Army's Ballistic Research Lab in Aberdeen MD programmed the ENIAC (Electronic Numerical Integrator and Computer) to determine π to 2037 digits – the first use of a computer to determine the digits of π. The reason given was "to determine π … with a view toward obtaining a statistical measure of the randomness of distribution of the digits" perhaps recalling DeMorgan's discovery of the fewer number of occurrences of the digit 7 in Shanks' erroneous determinations.

The ENIAC was not designed to perform such high precision calculations as the ENIAC only had 200 digits of decimal storage. Yet it was cleverly programmed to implement Machin's formula to perform a calculation that required more than 2000 digits of precision. It took 70 hours to perform the calculation mostly because intermediate results had to be punched out on IBM punch cards then later read back in. The calculation took place over the Labor Day weekend of 1949.

Aside: I became interested in exactly how the ENIAC with its peculiar instruction set and a store of 200 decimal digits could perform a calculation that required more than 2000 digits of precision. Researching how it could be done resulted in a publication: "The ENIAC's 1949 Determination of π"; IEEE Annals of the History of Computing ; Vol. 34, No. 3 (July – Sept. 2012). Hence my (vanity) license plate. The techniques were later used to write a Python program to determine digits of π to 10,000 digits of precisions – the first 1000 of which are displayed on page 85 with the poem.

My License Plate – Combining Computers & Mathematics

The ENIAC record for the determination of π held for 5 years. It was eventually surpassed in 1954 by the Navy's NORC (Naval Ordinance Research Calculator) computer's 13-minute computation of π to 3089 digits. In 1961 Daniel Shanks and John W. Wrench Jr. used an IBM 7090 computer to determine π to 100,00 digits in an 8-hour 43-minute run using the Machin-type arctangent formula

$$\pi = 24\arctan\frac{1}{8} + 8\arctan\frac{1}{37} + 4\arctan\frac{1}{239}$$

But the Machin-type arctangent-based formulas used are no longer enough.

5 - And beyond ...?

A new class of recursive algorithms was now needed and used to determine the digits of π out to millions and billions of digits. Even with faster and more powerful computers, the older Machin arctangent-based methods were computationally too slow; doubling the number digits upped the computational time by factor of 4 – it was a losing battle. If in 1961 100,000 digits required a little over 8.7 hours, 800,000 digits, well short of a million, would require more than 556.8 hours or 23.2 days. A rough estimate indicates over 8 years would be needed to achieve a goal of 1 billion digits. Of course, 21st century computers are a lot faster and more powerful than the 1960's IBM 7090 but it is still a losing battle.

And then there is Ramanujan and his formulas for π – see 1729

The current record for the digits of π (as of March 2019) is 31 trillion digits.

But this raises the question – why the interest in determining more and more digits of π?

Trivia Question: What is the millionth digit of π (after the decimal point) ? Answer: 1

The Basel Problem

Finally, a surprising result about π ! In 734 Leonard Euler proved $\sum_{k=1}^{\infty}\frac{1}{k^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+...=\frac{\pi^2}{6}$ a remarkable connection between the sum of the reciprocals of squared integers and π

Determining the sum of the *reciprocals of the squares* $\sum_{k=1}^{\infty}\frac{1}{k^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+...$ was a difficult problem. However, as it turned out, it had a very surprising solution! It was proposed by Jacob Bernoulli (1654 – 1705) who lived in Basel (hence the name the Basel Problem) and who was able to prove it converged to a value less than 2 but was unable to find an exact solution. Jacob's younger brother and mathematical rival Johann Bernoulli (1667 - 1748) was also unable to solve it as was Gottfried Wilhelm Leibniz! It was Leonard Euler (1707 – 1784) a student of Johann Bernoulli who in 1734 found the solution in one of the most beautiful and astonishing proofs in mathematics. The result is …

$$\sum_{k=1}^{\infty}\frac{1}{k^2}=1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\frac{1}{25}+...=\frac{\pi^2}{6}$$

Note that the sum of the reciprocals of the squared *integers* sums to a value containing the constant π, the circumference of a circle divided by its diameter (see π).

Finally: Monte Carlo Pi - Another Way to Calculate π - See The Truel

When π = 4?

In the July 1894 issue of the newly established American Mathematical Monthly, under the *Queries and Information* section, there appeared an article titled

"Quadrature of the Circle"
by Edward J. Goodwin, Solitude, Indiana

which was

"Published by the request of the author".

The opening paragraph of the article read

> "A circular area is equal to the square on a line equal to the quadrant of the circumference; and the area of a square is equal to the area of the circle whose circumference is equal to the perimeter of the square."

This was followed by the statement

"(Copyrighted by the author 1889. All right reserved)"

The article went on at some length about this new determination for finding the area of a circle but just going by the opening line, if the area of a circle is equal to the area of a square with sides equal to $\frac{C}{4}$, then $Area_{circle} = \left(\frac{C}{4}\right)^2$ as opposed to $Area_{circle} = \pi r^2$. Since by definition $\pi = \frac{C}{D} = \frac{C}{2r}$ or $C = 2\pi r$ it follows that $Area_{circle} = \pi r^2 = \left(\frac{C}{4}\right)^2 = \left(\frac{2\pi r}{4}\right)^2 = \frac{\pi^2 r^2}{4}$. Therefore

$$\pi r^2 = \frac{\pi^2 r^2}{4}$$
$$\frac{4\pi r^2 - \pi^2 r^2}{4} = 0$$
$$4\pi r^2 - \pi^2 r^2 = 0$$
$$\pi r^2 \cdot (4 - \pi) = 0$$
$$\pi = 4$$

In the defense of the American Mathematical Monthly, at that time the Monthly was privately published and apparently had a policy to publish articles at the request of the author (see above). In any case the issue might have ended there, except Dr. Goodwin (a physician) from Solitude in Posey County Indiana prevailed upon his local representative to the Indiana House, Taylor I. Record, to submit House Bill No. 246 in January 1897 that began

> "A bill for an act introducing a new mathematical truth and offered as a contribution to education to be used only in the state of Indiana free of cost by paying any royalties whatsoever on the same, provided it is accepted and adopted by official action in the legislature of 1897 …
>
> … Be it enacted by the General Assembly of the state of Indiana. That it has been found that a circular area is to the square on a line equal to the quadrant of the circumference as the area of an equilateral rectangle is to the square on one side". "

The passage of House Bill No. 246 through the Indiana Legislature makes for interesting and humorous reading. In defense of the Indiana legislators, from accounts I read, I got the impression that they knew the bill was bogus but had some fun playing around with it.

From the Indianapolis News, February 13, 1897, p. 11, col 3.

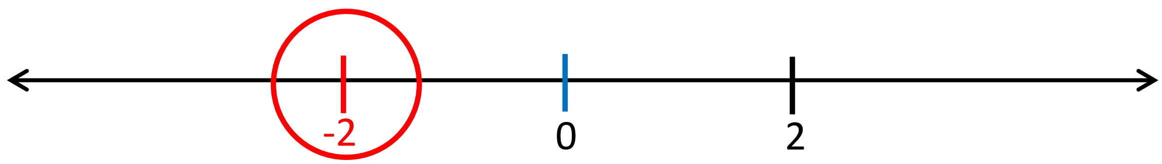
> "Representative Record's mathematical bill legalizing a formula for squaring the circle was brought up and made fun of. The senators made bad puns about it, ridiculed it and laughed over it. The fun lasted half an hour. Senator Hubbell said that it was not meet for the Senate, which was costing the state $250 a day, to waste time in such frivolity. He said that in reading the leading newspapers of Chicago and the East, he found that the Indiana State Legislature had laid itself open to ridicule by the action already taken on the bill, He thought consideration of such a proposition was not dignified or worthy of the Senate. He moved the indefinite postponement of the bill, and the motion carried".

It is interesting that Goodwin copyrighted his determination of the area of a circle as equal to "the square on a line equal to the quadrant of the circumference" and that he was willing to allow the state of Indiana to use his determination free of payment of royalties.

- 2

Grieved serfs' servitude for measured grain,
battling armies pushed back, homelands lost,
bankers' algorithms of debits/gains,
human number lines veer left towards cost.
My mind rambles up the lane to the faded old
apartment, the frail gray pair of them smiling,
inviting me for tea, in from the cold,
to listen to stories, play silly games and sing.
The sum of them and their unabashed love
an absolute value more than they themselves.
At notice of their deaths, I shirked the move
through negative numbers, mumbled a farewell.
Years on, remembrances that I brushed off
in forfeiture still extract gold tears of loss.

-- E R Lutken (3: A Taos Press © 2021)



Baucis and Philemon

The Roman writer Ovid's (43 BCE – 17 or 18 CE) collection of Greek/Roman fables relates the story of Baucis and Philemon, an older couple who welcomed a disguised Zeus and Hermes to their humble cottage after the two travelers had been turned away by others in the area. Exacting punishment on those who refused hospitality to strangers (a breach of social standards), Zeus and Hermes destroyed the area but spared Baucis and Philemon turning their humble cottage into an ornate temple. The older couples' wish to be guardians of the temple was granted along with their wish that when one of them died, the other would at the same time; thus, they were joined together in death as two trees, an oak and a linden.

10_2 = 2: Base 2 Binary Numbers

One of the advantages of our modern positional notation for representing numbers is that numbers using a different base can be easily represented.

In binary or base-2 notation, the weight of each binary digit a.k.a. *bit* (either 0 or 1) from *right to left* is a power of 2 instead of a power of 10 as in our commonly used decimal positional notation. So

$$101101 = 1\times 2^5 + 0\times 2^4 + 1\times 2^3 + 1\times 2^2 + 0\times 2^1 + 1\times 2^0 = 32 + 8 + 4 + 1 = 45$$

The powers of 2 are easy to compute: 1, 2, 4, 8, 16, 32, 64, 128, … etc. just double the previous value which makes converting from *binary to decimal* fairly easy to do by *summing the corresponding power of 2* indicated by the 1's digits as done above.

Converting *decimal to binary* is a bit more trickly but can be done by using the following tableau to subtract (and record) the corresponding powers of 2.

Example: Convert 45 (decimal) to binary using the following tableau form

0						
64	**32**	**16**	**8**	**4**	**2**	**1**
	45					

That is, write out the power of 2 right to left *in the second row* until you exceed the value being converted (64 > 45). Put a 0 in the row above (indicating there are no 64's in 45).

0	1	0	1	1	0	1
64	**32**	**16**	**8**	**4**	**2**	**1**
	45 -32 --- 13		13 -8 --- 5	5 -4 --- 1		1 -1 --- 0

Next subtract the next smallest power of 2 from the number being converted, placing a 1 in the row above that power of 2. Continue working *left to right* subtracting the corresponding power of 2 from the remaining difference and putting a 1 in the row above the corresponding power of 2 if subtraction is possible or putting a 0 in the row above the corresponding power of 2 if subtraction is not possible. Continue down to $1 = 2^0$ at which point the first row will have the corresponding binary representation (leading 0's are quite ok).

Aside: The above conversion algorithms work best with paper and pencil; there are other algorithms which are easy to program.

Advantages and Disadvantages of Binary Numbers

There are two advantages to binary numbers. First, 2-state electronic components are cheap and easy to produce (vs ten-state electronic components) which is why computers use binary notation internally. Second and perhaps more importantly, algebraic operations like addition, subtraction, multiplication, division, and yes even square roots, are easier to do in binary than in decimal – a fact that carries over to designing the corresponding electronic circuits (see *Math History in a Few Bad Clerihews*).

The big disadvantage is that you need a lot of binary digits (bits) to represent even medium sized numbers. An *n* digit decimal integer requires on the order of $3.3n$ bits. For example, 10-digit decimal number requires approximately 33-36 bits. 999 decimal (3 digits) is 11111 00111 binary (10 bits).

Doing Arithmetic in Binary

The rules for binary addition and binary subtraction are presented below with and without a carry in from the right (addition) and with and without a borrow from the right (with subtraction). Four of the eight addition rules have a carry out to the left and four of the eight subtraction rules have a borrow from the left. Four of the eight addition rules have a carry in from the right and four of the eight subtraction rules have a borrow from the right.

Addition Rules (with and without a *carry out* to the left indicated by **1** and with and without a *carry in* from the right indicated by **+1**)

```
                                         +1    +1    +1    +1   <- carry in
  0     1     0     1              0     1     0     1
 +0    +0    +1    +1             +0    +0    +1    +1
 --    --    --    --             --    --    --    --
  0     1     1    10              1    10    10    11
```

Subtraction Rules (with and without a borrow from the left denoted by **-1** and with and without a borrow from the right indicated by **-1**).

```
                                       -1       -1       -1       -1   <- borrow
  0       1        0       1          0        1        0        1
 -0      -0       -1      -1         -0       -0       -1       -1
 --      --       --      --         --       --       --       --
  0       1    -1  1       0      -1  1        0    -1  0    -1  1
```

The rules for subtraction can be somewhat confusing when compared to the addition rules. For example, the third rule subtracting 1 from 0 requires borrowing 2 units from the left. Subtracting 1 from the 2 borrowed units leaves a difference of 1 with the indicated *borrow from the left* (**-1**).

The seventh rule subtracting 1 from 0 with a borrow from the right (-1) requires borrowing 2 units from the left to cover both the 1 being subtracted and the 1 borrowed from the right. Hence the difference is 0 and the borrow is propagated to the left (-1).

Multiplication Rules: The rules for multiplication are simpler; if the single digit multiplier is 1, you simply copy the multiplicand; otherwise with a 0 multiplier the product is zero.

```
  0     1     0     1
 ×0    ×0    ×1    ×1
 --    --    --    --
  0     0     0     1
```

```
   1101
  × 101
-------
   1101
  0000
 1101
-------
1000001
```

Computing partial products for multi-digit multiplicands and multipliers (see example on the left) is greatly simplified: If the multiplier digit is 1, the partial product is just the multiplicand (appropriately left shifted). If the multiplier digit is 0, the partial product is all zeros (appropriately left shifted). The complexity for multi-digit binary multiplication is in summing the partial products. Recall that multiplication *is* repeated addition,

Division Rules: You can't divide by 0 and anything divided by 1 is anything. However *long division* where you have a multi-digit dividend and, a multi-digit divisor is a different story!

```
           1010 r 101
       ----------
110 / 1000001
      - 110
        ----
          10↓
          100↓
          1000
        - 110
          ----
            10↓
            101
```

In the long division example on the left we're dividing 65 = 1000001_2 by 6 = 110_2 using the standard paper and pencil long division approach.

110_2 *goes into* 1000_2 the *partial dividend* 1 time (1 is the first quotient digit) so subtracting we have 10_2 left over.

Bringing down the next zero (denoted by ↓), 100_2 is smaller than 110_2 so 0 is the next quotient digit.

Bringing down the next zero is 1000_2 . We can subtract 110_2 . 110_2 *goes into* 1000_2 1 times (1 is next quotient digit) with 10_2 left over.

Bringing down the final 1, 101_2 is still too small so 0 in the final quotient digit and we have 101_2 as a remainder. One advantage to long division in binary is that the quotient digit is either 0 or 1 depending on whether the divisor is less than or greater than or equal to the partial dividend. A simple comparison is all that is needed unlike decimal long division where you *guestimate* how many times the divisor *goes into* the partial dividend (and then multiply and subtract to get the next partial dividend). Note $65 \div 6 = 10 \ rem \ 5$

Observe that the complexity for binary long division is due to repeated subtraction!

Computers and Binary Representation

As mentioned above computers represent numbers internally in binary since bi-stable electronic components are cheap and circuits to do arithmetic operations like addition, subtraction etc. in binary are less complicated.

Computers store binary integers using a fixed number of bits. A *byte* is 8-bits; 4 bytes = 32 bits is a *word*, and 8 bytes = 64 bits is a *double word*. Obviously, the more bits used to store an integer the greater the range of integers that can be represented. Today the default standard seems to be 64-bit integers (*double words*!).

Restricting the size of integers to 8 bits (binary digits) gives us a range from 0000 0000 equaling 0 (the smallest representable integer) to 1111 1111 equaling 255 (the largest positive integer).

```
0000 0000 | 0000 0001 | 0000 0010|  . . .  |1111 1101 | 1111 1110 | 1111 1111
    0           1           2                  253         254         255
```

Representing Negative Binary Numbers
n-bit Twos Complement Binary Notation

What about negative binary integers? How does one represent **-2**?

A standard approach for representing signed binary integers where the number of bits is *fixed* is to use the left most or most significant bit as the sign bit using 0 for plus and 1 for minus.

With the *sign-magnitude* technique the left-most bit is simply the sign – so 0000 1101 is plus 13 while 1000 1101 is minus 13. 8-bits gives a range of -127 to +127. Unfortunately, with this method there are two zeros: 0000 0000 or +zero and 1000 0000 or minus zero.

Aside: Mathematically speaking zero is neither positive nor negative.

A better method and the one used today is called *n-bit twos complement notation* where the sign bit (the left-most bit) is given a negative value or *weight*. In the case of 8-bit two complement notation the *weight* of the sign bit is $-2^7 = -128$ leaving the rest of the bits to keep their positive weights.

For example:

1111 1110 =
$-1\times2^7 + 1\times2^6 + 1\times2^5 +1\times2^4 + 1\times2^3 + 1\times2^2 + 1\times2^7 + 0\times2^0 =$
$-128 + 64 +32 + 16 + 8 +4 + 2 = -2$

And of course, 0000 0010 = +2 (leading zeros!)

With 8-bit twos complement notation, the range is -128 … +127

```
1000 0000 | 1000 0001 | 1000 0010 | ... | 1111 1101 | 1111 1110 | 1111 1111
  -128        -127        -126                -3          -2          -1

0000 0000 | 0000 0001 | 0000 0010 | ... | 0111 1101 | 0111 1110 | 0111 1111
    0           1           2                125         126         127
```

One interesting side effect of n-bit twos complement representation is that zero is a positive number since its sign bit is 0 (positive). As mentioned above, this is an interesting difference between mathematics and computer science since mathematically, zero is neither positive nor negative.

Aside: The hardware of a computer can detect the sign of a number by simply examining the most significand bit; a very simple circuit to implement.

Doing Arithmetic in n-bit Twos Complement Binary Representation

Addition: The standard eight rules for binary addition hold for n-bit twos complement binary representation.

Negating: Negating an n-bit two complement binary integer is easy: to negate, you *bitwise complement* each bit then add 1.

Example: To negate +45 = 00101101

```
 00101101
 11010010  ← bitwise complement
+       1    plus 1
 --------
11010011 = -128 + 64 + 16 + 2 + 1 = -128 + 83 = -45
```

And if we negate -45 (bitwise complement and add 1) …

```
      11010011
      00101100 ← bitwise complement
     +       1   plus 1
      --------
      00101101 = +45
```

And we're back!

Subtraction: Subtraction is now done by negating the subtrahend (easily done by complement and add +1) and ***adding*** it to the minuend with any carry out from left-most position thrown away. For example,

```
 45     00101101 ----------------->  00101101
-22    -00010110 -----> 00010110    +11101010
 --     --------        11101001     --------
                       +       1     00010111 = 23
                        --------
                       11101010 = -23

 22    00010110 ----------------->   00010110
-45   -00101101 -----> 00101101     +11010011
 --    --------        11010010      --------
                      +       1      11101001 = -128+64+32+8+1 = -23
                       --------
                       11010011 = -45
```

Thus, there is no need for a separate subtraction circuit or separate subtraction rules. Furthermore, *by hand* addition is easier to do than subtraction (recall the 8 subtraction rules given above). Of course, negative results are now possible if the subtrahend is larger than the minuend.

Carpenter's Song

On the job, like a good greasy
ham and cheese sandwich,
not some lettuce bowl that needs a fork
twelve is the best number.

Rulers, yardsticks, squares are decimal sizes,
and twelve divides into halves, fourths,
into thirds, sixths and, of course twelfths.
Can't beat it for building
twelve is the best number.

Centimeters may be fine for measuring
the length of a pill, or calculating a trip
to Mars, where all those conversions can
skip through your head in a stream of zeros,
but for work you lift with your hands,
pull out a tape and mark with a fat pencil,
twelve is the best number.

-- E R Lutken (3: A Taos Press © 2021)

Why 12?

Base ten we get (?) from our fingers and thumbs – so decimal notation is in some ways understandable. Of course, the ancient Babylonians used base 60 (sexagesimal) vestiges of which have come down to us today as 60 seconds in a minute, 60 minutes in an hour, and 360 degrees in a full circle. And it is true that 60 (like 12) is highly divisible - by 2, 3, 4, 5, 6, 10, 12, 15, 20, and 30, but unlike our modern decimal positional notation (inherited from India) with ten symbols 0 thru 9 to represent digits, the Babylonians repeated symbols for ten and one to represent values between 1 and 59.

There is an interesting theory that the high divisibility of 60 was not the reason base 60 was used. The 12 finger phalanges – the three joints of the four fingers (phalanges) when touched by the opposing thumb could be used to count. When a full count of 12 was reached, a finger on the other hand could be extended to mark a group of 12 and the other hand reset. A full count of four extended fingers (4 × 12) on one hand plus 12 phalanges on the other yielded 60. In fact, one could actually do addition by counting on one's fingers.

Though twelve might be the *best number*, we don't have a base 12 number system – to do so would require adding two new digits – 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, T (for ten), E, (eleven) or maybe taking a hint from hexadecimal (base 16) notation 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, A, B.

So, which is the best number base?

Base 2, 2^3=8, 2^4=16?

Modern computers use binary notation – base 2 with the digits 0 and 1 (see entry for -*2*). This was done for two reasons. First, bi-stable electronic circuits are cheap and easy to fabricate and second, arithmetic operations done in binary are less complicated to do. For example, consider the size of a ten by ten decimal multiplication table compared to a two by two binary multiplication table

Decimal Multiplication Table										
×	0	1	2	3	4	5	6	7	8	9
0	0	0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7	8	9
2	0	2	4	6	8	10	12	14	16	18
3	0	3	6	9	12	15	18	21	24	27
4	0	4	8	12	16	20	24	28	32	36
5	0	5	10	15	20	25	30	35	40	45
6	0	6	12	18	24	30	36	42	48	54
7	0	7	14	21	28	35	42	49	56	63
8	0	8	16	24	32	40	48	56	64	72
9	0	9	19	27	36	45	54	63	72	81

Binary Multiplication Table		
×	0	1
0	0	0
1	0	1

Of course, the downside of binary is that lots of 0's and 1's are needed for representing even moderately sized numbers.

To partially address this problem with binary notation, octal (base 8) and hexadecimal (base 16) notations were introduced since it is easy to convert between binary and octal/hexadecimal using a simple *group by three* for octal and *group by four* for hexadecimal conversions. To represent large numbers, octal and hexadecimal notations need fewer digits than binary (so were considered more *human-friendly*). However more *digits* are needed for hexadecimal.

The letter A through F are used in hexadecimal to represent the decimal values 10 through 15.

decimal	binary	octal	hexadecimal	decimal	binary	octal	hexadecimal
0	0000	0	0	8	1000	10	8
1	0001	1	1	9	1001	11	9
2	0010	2	2	10	1010	12	A
3	0011	3	3	11	1011	13	B
4	0100	4	4	12	1100	14	C
5	0101	5	5	13	1101	15	D
6	0110	6	6	14	1110	16	E
7	0111	7	7	15	1111	17	F

Any sufficiently advanced technology is indistinguishable from magic.
-- Arthur C. Clark's 3rd Law

Any sufficiently advanced civilization uses a numeric notational system based on a power of 2.
-- Shelburne's corollary (with all modesty)

Biquinary Notation and Roman Numerals

Biquinary notion represents a "decimal" digit (or value) as a pair – usually a five and a one. For example, eight would be a "five + three ones". Roman Numerals are similar to but not quite biquinary since they use different symbols for five (V), 50 (L), and 500 (D) as well as different symbols for one (I), ten (X), 100 (C), and 1000 (M). The Roman Numeral VIII equals eight while the Roman Numeral LXXX equals 80.

Is there a best base for numbers?

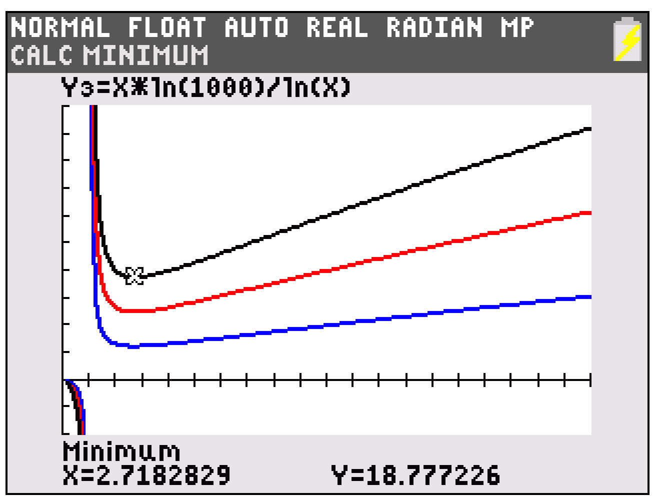
Binary notation requires only two digits: 0 and1 but you need a lot of digits (or bits) to represent even a moderately sized number (1000 decimal requires 10 bits: 1111101000_b). Decimal notation has ten digits, but fewer digits are needed to represent moderately sized numbers. Is there a *best base* or most efficient base between 2 and 10? That is, given the trade-off between the number of digits needed for a given base β and the number of digits needed to represent an integer *n* is there a *best base*?

> Observe that given any integer n, the number of digits base β needed to represent integer *n* is given by $\frac{\ln(n)}{\ln(\beta)}$ where $\ln(x)$ is the natural log function (recall *Phaetons' Ride* and *Measured Illusion*). For example, and you can compute this yourself, for base *10*, 1000 (decimal) requires $\frac{\ln(1000)}{\ln(10)} = 3$ decimal digits while in base *2* the same value requires, $\frac{\ln(1000)}{\ln(2)} = 9.9657$ rounded up to 10 binary digits. Actually it doesn't matter what base logarithm function you use, $\frac{\log_{10}(1000)}{\log_{10}(10)} = 3$ and $\frac{\log_{10}(1000)}{\log_{10}(2)} = 9.9657$ where $\log_{10}(x)$ usually just written as $\log(x)$, is the log base 10 or *common logarithm*.

So the larger the base β, the smaller the number of digits $\frac{\ln(n)}{\ln(\beta)}$ needed to represent an integer *n* while the smaller the base β, the larger the number of digits $\frac{\ln(n)}{\ln(\beta)}$ needed for *n*. As one increases (either β or *n*) the other (either *n* or β) decreases. This suggests that the product of base β with $\frac{\log_e(n)}{\log_e(\beta)}$ to form the function $f(\beta) = \beta \cdot \frac{\ln(n)}{\ln(\beta)}$ might have a *minimal value* for some base β. And indeed, it does – *independent of n*.

If we graph $f(\beta) = \beta \cdot \frac{\ln(n)}{\ln(\beta)}$ for different values of n (see plot on right) we see that there is value for β which minimizes the function

$$f(\beta) = \beta \cdot \frac{\ln(n)}{\ln(\beta)}.$$



In the plot on the right, we plotted $f(\beta) = \beta \cdot \frac{\ln(n)}{\ln(\beta)}$ for three different values of n: 10 (blue), 100 (red) and 1000 (black). Note how all three graphs appear to have the same minimum value at 2.7182829 which is *e*.

Aside: We can also find this minimum point using calculus by taking the derivative of $f(\beta)$, solving for zero, and testing that the critical point is absolute minimum (recall *Zero*).

So, the most efficient base is base *e = 2.7182818…* which is not an integer. The closest integer is 3 which leads us to the next consideration…

Ternary Notation

If binary notation is base 2, ternary notation is base 3 with three digits 0, 1, and 2 where the weight of each digit is a power of 3. For example:

$$1201 = 1\times 3^3 + 2\times 3^2 + 0\times 3^1 + 1\times 3^0 = 27 + 2\times 9 + 1 = 46$$

Thus, it's easy to do *ternary to decimal conversion* by expanding and adding the powers of three with the non-zero coefficients.

For example: $12_5 = 1\times 3^1 + 2\times 3^0 = 3 + 2 = 5$ and $122_3 = 1\times 3^2 + 2\times 3^1 + 2\times 3^0 = 9 + 6 + 2 = 17$

Postfix subscripts *are used to indicate the base or radix of the number for non-decimal representation.*

Converting decimal to ternary is a bit harder as it's done by subtracting out powers of 3. This can be formally done using a variation of the tableau method for decimal to binary conversions (recall *-2*) where we use and list the powers of 3 (e.g. 81 27 9 3 1) right to left in the second row.)

Example: converting 46 to ternary notation:

0	1	2	0	1
81	27	9	3	1
	46 -27 19	19 -18 1		1 -1 0

Check: $1201_3 = 1\times 3^3 + 2\times 3^2 + 0\times 3^1 + 1\times 3^0 = 1\times 27 + 2\times 9 + 0\times 3 + 1\times 1 = 46$

Obviously, it helps to know the powers of 3: 1, 3, 9, 27, 81, 243, 729 …

Another technique is to repeatedly divide by 3 and note the remainders. This gives you the ternary digits in reverse order (from least significant to most significant).

```
3 \ 46
   ----
3 \ 15    r 1
   ----
3 \  5    r 0
   ----
3 \ 1     r 2
   ----
    0     r 1           1 2 0 1
```

Aside: The same technique where we divide by 2 and note the remainders can be used to convert decimal to binary.

Balanced Ternary Notation – The Goldilocks of Numbers

If base 3 (ternary notation) uses the digits 0, 1, and 2 then *balanced ternary notation* uses the digits 1, 0 and -1 (which could be written as *(-1)*) . Since a digit can have a negative value, -1 or (-1) we'll use the digit M for (-1) since it's "notationally" awkward to use (-1).

For example, $5 = 1MM = 1\times3^1 + (-1)\times3^1 + (-1)\times3^0 = 9-3-1=5$. While at first glance balanced ternary seems awkward, it has several interesting features.

First, positive numbers begin with 1, negative with M. For example: $M01_3 = (-9)+1 = -8$ while $10M_3 = 9+(-1) = 8$. Not need for separate + and – signs.

Second, to negate exchange M's with 1's and vice versa leaving 0's alone. Negating $M01_3$ is $10M_3$.

Third, balanced ternary addition rules are given in the following table.

Balanced Ternary Addition			
+	M	0	1
M	M1	M	0
0	M	0	1
1	0	1	1M

Of the nine addition rules, only two require a carry: M + M = M1 and 1 + 1 = 1M. There are two *cancelation rules:* M+1=0 and 1+M=0. The other five rules have 0 as one of the operands (addition with 0) . This simplifies addition.

Balanced ternary to decimal conversion is done by expanding by powers of powers of 3 (allowing negative values) as was done with ternary to decimal notation. For example:

$$1M01 = 1\times3^3 + (-1)\times3^2 + 0\times3^1 + 1\times3^0 = 27-9+1=19$$

Decimal to balanced ternary conversion of positive integers is a two-step process. First convert to decimal to ternary.

$$19 = 201_3$$

Next *replace all 2's with M and add 1 to the next digit up* (a carry?) using the rules for addition (which may require some propagation of carries)

$$201_3 = 1M01_3$$

For a negative decimal integer, convert the positive absolute value to balanced ternary then negate.

Balanced Ternary Arithmetic

Addition was covered above. Subtraction is done by first *negating the subtrahend* then *adding it to the minuend* Remember to negate a value, replace1 by M and M by 1 (leaving 0 alone). For example:

```
  12      110      110                     7      1M1       1M1
 - 7    - 1M1    + M1M                   -12    - 110     + MM0
 ---     ----     ----                   ---      ---       ---
   5               1MM                    -5                M11
```

Multiplication is also easily done since each partial product is either the multiplicand properly shifted if the corresponding multiplier digit is 1, 0 if the multiplier digit is 0, or the negative multiplicand (easily computed) properly shifted if the multiplier digit is M. Again, the complexity is in summing the partial products.

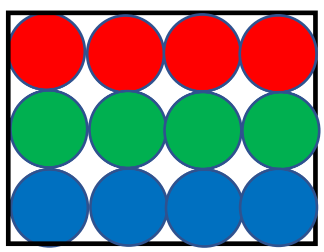
Balanced Ternary Multiplication			
×	M	0	1
M	1	0	M
0	0	0	0
1	M	0	1

Division is left as an exercise for the student. (Hint: Recall binary division covered in -*2*.)

Which is the best base?

"Rulers, yardsticks, squares are decimal sizes,
and twelve divides into halves, fourths,
into thirds, sixths and, of course twelfths.
Can't beat it for building
twelve is the best number."

However, from a purely human point of view the fact that twelve easily divides into halves, thirds, fourths, and sixths (common fractions) does make 12 useful. Besides canned goods pack nicely into 3 x 4 or 4 x 6 (24 items) arrays for boxing and a dozen eggs pack nicely into 2 x 6 cartons.

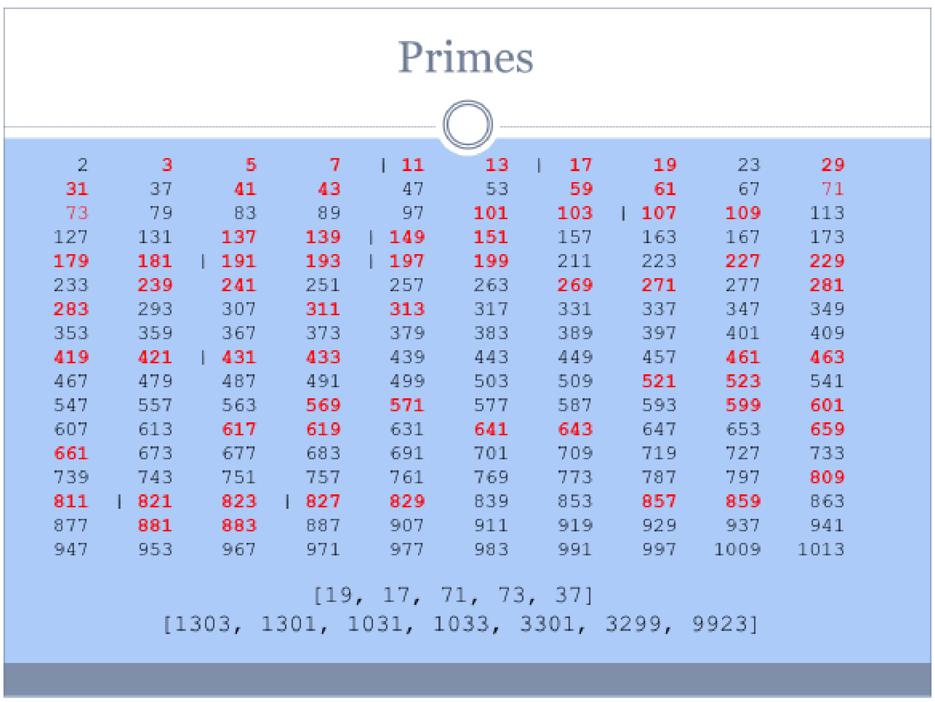


The base 10 metric system never really caught on in the United States

Prime Syllable Song

Prime
numbers
do not care
to make a rhyme.
They blaze wild pathways.
While others tow the line,
they play crwths or smash guitars,
unlike composite counts assigned
to echo harmonic notes in time
and avoid the oddball and ill-defined.
Without primes, though, the music would be boring,
the sing-song regularity, the constant whine
would drive all of us absolutely batshit bonkers. – E.R. Lutken (3: A Taos Press © 2021)

Results About Primes



Primes vs Composite Numbers

A *prime* is an integer greater than 1 whose only divisors are itself and 1 – for example 17. A number which is not prime is called a *composite* number. All composite numbers have prime divisors (aside from itself and 1). For example, the composite number 15 has 3 and 5 as prime divisors.

Note that 1 is considered to be *neither* prime *nor* composite.

These are 25 primes less than 100. The ones in red are *twin primes*, pairs of primes which differ by 2.

2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37, 41, 43, 47, 53, 59, 61, 67, 71, 73, 79, 83, 89, 97

There are 168 prime numbers less than 1000.

The first fact about primes is that every integer greater than 1 has a prime divisor. Seems obvious but here is the proof which depends on something called the Well-Ordered Property of Positive Integers.

Well Ordered Property: Every non-empty set of positive integers has a least element.

This property of positive integers should be intuitively obvious so it requires no proof; it's axiomatic! The positive integers are ordered and given any non-empty set of them, they can be arranged from smallest to largest with the smallest being the least element.

Proving that every integer has a prime divisor makes use of an indirect proof (recall *Irrational Loss*) where assuming the conclusion is *false* leads to a *contradiction* forcing us accept the conclusion as being *true*.

Let's assume the conclusion is false: there are positive integers greater than 1 which have no prime divisor. Let's call this non-empty set of positive integers greater than 1 with no prime divisors S. By the Well-Ordered Property set S has a least element, call it *k*. Obviously *k* is not prime since each prime is its own divisor and *k* >1, so *k* is composite meaning that *k* has at least two smaller factors *a* and *b* both greater than 1 such that $k = a \times b$. But *a* and *b* are not in set S (*k* is the smallest element in S) so both *a* and *b* have prime factors which by the *transitivity of division* (if x divides y and y divide z then x divides z) implies *k* has prime factors obtained from *a* or *b*. So *k* cannot be in S so a *least element k* for at non-empty set S cannot exist. For the Well-Ordered Property to be true, the set S must be empty. Therefore, every positive integer has a prime factor! - QED

Is there a largest prime?

There is no largest prime – a result and its proof found in Euclid's *Elements* ca. 300 BCE. The proof is quite simple and uses the following result.

Result: Given any finite set of primes, there is another prime not in that set!

Proof: Given any finite set of *n* primes $\{p_1, p_2, p_3, \ldots p_n\}$ consider the number p obtained by multiplying the *n* primes together plus 1: that is $p = p_1 \times p_2 \times p_3 \times \ldots \times p_n + 1$. Now either p is prime (so *p* is a prime not in the set) OR p has a prime factor q but in this latter case q cannot be one of the p_i for $i = 1, 2, 3, \ldots, n$ because if $q = p_k$ for some p_k in the set $\{p_1, p_2, p_3, \ldots p_n\}$ then $q = p_k$ divides the difference $p - p_1 \times p_2 \times p_3 \times \ldots \times p_n = 1$ so q would divide 1 which is impossible!

Corollary: The set of primes is infinite since if the set of primes was finite, using the above result we can find an additional prime not in the set.

However, as we look at larger and larger integers the primes seem to become less numerous. In fact,

there are arbitrarily long stretches of consecutive integers which contain no primes.

For example. consider the product of the consecutive primes from 2 to 11; that is …

$$2\times3\times5\times7\times11=2310.$$

The 11 consecutive integers 2312, 2313, …, 2322 obtained by adding the eleven consecutive integers 2, 3, … 12 to 2310 are all composite (why?).

Hint: 2312 = 2310+2, 2313 = 2310+3, 2314 = 2310+4 ,…, 2321 = 2310+11, 2322 = 2310+12

To generalize: If p_n is the n^th^ prime then adding the p_n consecutive integers 2 through p_n+1 to the product of the consecutive primes 2 through p_n results in a stretch of p_n consecutive integers all of which are composite.

For example, if p_n is a prime greater than a million, there is a stretch of more than a million consecutive integers all of which are composite – no primes.

So, while the primes form an infinite set, there are arbitrarily long stretches of integers with no primes!

Primes as building blocks of the integers

Primes are often referred to as the *building blocks of the positive integers* since every integer greater than 1 can be uniquely factored into a product of primes. This is the so-called *Fundamental Theorem of Arithmetic* which states: Every positive integer greater than one can be *uniquely* decomposed into a product of primes – a result and proof also found in Euclid's *Elements* (300 BCE).

For example: $60=2\cdot2\cdot3\cdot5=2^2\cdot3\cdot5$ or $210=2\times3\times5\times7$

This also supports the idea of why 1 should not be considered a prime since the *uniqueness* of the factorization would be compromised by allowing none or multiple factors of 1 to be included in a product

"Without primes, though, the music would be boring,"

Finding primes

It's not difficult to determine if a positive integer n is prime or composite. Since a composite n must have a prime divisor smaller than $\sqrt{n}$, we only need to test if n is divisible by some prime less than $\sqrt{n}$.

For example, any integer less than 100 has a prime divisor less than √100 = 10. There being only 4 primes less than 10, that is 2, 3, 5, and 7, a simple test for primality is to *test divide n* by each of 2, 3, 5, and 7. If none of the test divisors (2, 3,5, and 7) divide *evenly* into *n* (i.e. each division has a non-zero remainder), then *n* is prime.

Aside: Checking for divisibility by 2, 3 and 5 is easy since there are simple rules for each; checking for divisibility by 7 is a bit harder.

Since the eleven primes 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, and 31 are all less than $\sqrt{1000} = 31.622$ (with 37 being the next largest prime) at most 11 trial divisions will detect any prime less and or equal to 1000.

And given the 25 primes less than 100,

```
2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,53,59,61,67,71,73,79,83,89,97
```

at most 25 trial divisions will detect a prime less than $100^2 = 10,000$

See the Appendix for a Python Program that uses 25 primes less than 100 to detect primes up to 10,000.

A more efficient test for primes is based on the converse of the **Fermat Little Theorem** discussed below BUT it's doesn't always work!

How are the primes distributed?

As mentioned above, the number of primes is infinite; there is no largest prime. However, as we go further out on the number line of integers, the primes become more scarce. Moreover, as shown above, there are arbitrarily long sequences of consecutive integers where no primes are to be found.

How are the primes distributed? That is, is there a way to determine the number of primes less than or equal to some integer *n*? We define the *prime counting function* $\pi(n)$ to be the number of primes less than or equal to *n*. (Note $\pi(n)$ is not to be confused with the constant π)

For example, $\pi(10) = 4$ since there are 4 primes (2,3,5,7) less than or equal to 10. $\pi(11) = 5$, $\pi(12) = 5$ while $\pi(13) = 6$. The value of $\pi(n)$ *jumps* when *n* is a prime number.

Parenthetically, $\pi(n)$ could be used to test if an integer *n* is prime since *n* is prime if and only if $\pi(n) = \pi(n-1) + 1$; that is there is a *jump* at *n*. Unfortunately, there is no "exact equation" for $\pi(n)$. We can compute $\pi(n)$ for *small* values of n by listing and counting all primes less than or equal to *n* but this is not an efficient method. For example …

n	$\pi(n)$
10	4
100	25
1,000	168
10,000	1229
100,000	9592
1,000,000	78498

However, the Prime Number Theorem states:

$$\lim_{n\to\infty} \frac{\pi(n)}{(n/\ln(n))} = 1$$

That is *in the limit* as *n* get large the number of primes less than or equal to *n* is approximately $\frac{n}{\ln(n)}$.

The key idea is *in the limit* as *n* approaches infinity; otherwise, it doesn't seem very accurate.

For example: $\pi(100) = 25$ while $\frac{100}{\ln(100)} \approx 21.71$: ratio = $\frac{25}{21.71} \approx 1.1515$

$\pi(1000) = 168$ and $\frac{1000}{\ln(1000)} \approx 144.76$: ratio $\frac{168}{144.76} \approx 1.1605$

$\pi(10,000) = 1229$ and $\frac{10,000}{\ln(10,000)} \approx 1085.74$: ratio $\frac{1229}{1085.74} \approx 1.1319$

$\pi(100,000) = 9592$ and $\frac{100,000}{\ln(100,000)} \approx 8685.89$: ratio $\frac{9592}{8685.89} \approx 1.1043$

$\pi(1,000,000) = 78,498$ and $\frac{1,000.000}{\ln(1,000,000)} \approx 72382.41$: ratio $\frac{78498}{72382.41} \approx 1.0845$

"Prime
numbers
do not care
to make a rhyme.
They blaze wild pathways."

So although the approximation to $\pi(n)$ given by $\frac{n}{\ln(n)}$ from the Prime Number Theorem does not seem very accurate, if we take the *limit of the quotient* as n get large, the ratio does approach 1.

It is interesting that Carl Friedrich Gauss (1777-1855) suggested an alternate approximation to $\pi(n)$, the logarithmic integral function $Li(n) = \int_2^n \frac{1}{\ln(t)} dt$ which seems to give a slightly more accurate approximation. For example, $Li(1000) = 176.5645$ (recall $\pi(1000) = 168$).

Some simple unknown results about primes

There are some *simple to state* but *no proof of* statements about primes; that is *conjectures* about primes.

Goldbach's Conjecture

Is every even integer n $\geq$ 4 the sum of two primes? e.g. 100 = 41 + 59. In 1742, Christian Goldbach (1690 – 1764) communicated this observation to the famous mathematician Leonard Euler (1707 – 1783). While easy to demonstrate, it has not been proved (or disproved as no counterexample has been found). Nevertheless, many, including Euler, suspect it to be true (but perhaps unprovable?).

The Twin Prime Conjecture

Twin Primes (in *red*) like 3 & 5, 5 & 7, 11 & 13, 17 & 19, 29 & 3, 41 & 43, 59 & 61, 71 & 73 etc. are primes that differ by 2. (Note 3, 5, 7 is the only triple prime set.) An interesting feature is that if *p* and *p+2* are twin primes, then *p+1* is divisible by 6. For example, 71 and 73 are twin primes and 72 is divisible by 6. However, while 24 is divisible by 6, only 23 is prime (the smallest *singleton* prime). There are eight pairs of twin primes less than 100 (listed above), 35 twin prime pairs less than 1000, and 205 twin prime pairs less than 10,000 as compared to 25 primes less than 100, 168 primes less than 1000 and 1229 primes less than 10,000. Not surprisingly and like the primes, twin primes get scarcer as the integers increase. However, while we know there is no largest prime (the number of primes is countably infinite), what we don't know is whether the same is true for the twin primes even though their numbers decrease faster as we go out the number line of integers.

The *Twin Prime Conjecture* says there is no largest pair of twin primes: easy to state and understand; obviously difficult (or impossible?) to prove.

Beyond Twin Primes

Cousin Primes? These are prime pairs that differ by 4. For example: 3 & 7, 7 & 11, 13 & 17, 19 & 23, 37 & 41, 43 & 47, 67 & 71, 79 & 83, 97 & 101 – 9 pairs less than 101. There are 41 pairs under 1000

Sexy Primes? These are prime pairs that differ by 6. For example: 5 & 11, 7 & 13, 11 & 17, 17 & 23, etc. Sexy is a "play" on the Latin word for six (6): *sex*. There are 14 pairs less than 103 (97 & 103 being the largest) and 74 pairs less than 1000

Summing the reciprocals of primes

Consider the integers: 1, 2, 3, 4, 5, … and their reciprocals $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots\ldots$

The Harmonic Series $\sum_{k=1}^{\infty} \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \ldots + \frac{1}{k} + \ \to \infty$ diverges (recall *Measured Illusion*)

Consider the powers of 2: 1, 2, 4, 8, 16, …, 2^n, … and their reciprocals $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots, \frac{1}{2^n}, \ldots$

The Sum of the Reciprocals of the powers of 2 $\sum_{k=0}^{\infty} \frac{1}{2^k} = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots + \frac{1}{2^k} + \ldots = 2$ converges (recall *Cantor's Ghazal*)

Consider the squares: 1, 4, 9, 16, 25, … and their reciprocals $1, \frac{1}{4}, \frac{1}{9}, \frac{1}{16}, \frac{1}{25}, \ldots$

The Sum of the Reciprocal Squares $\sum_{k=1}^{\infty} \frac{1}{k^2} = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \ldots + \frac{1}{k^2} + \ldots = \frac{\pi^2}{6}$ converges – a famous result (the Basel Problem) proved by Leonard Euler (1707-1783) (see π).

Consider the primes: 2, 3, 5, 7, 11, 13, … and their reciprocals $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}, \frac{1}{11}, \frac{1}{13}, ...$

The Sum of the Reciprocal Primes: $\sum_{p_k \in P} \frac{1}{k} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + ... + \frac{1}{p_k} + ... \to \infty$ diverges where p_k is the k[th] prime from the set of all primes P – again proved by Leonard Euler. .

Somehow, the *gaps between consecutive primes* are, in some sense, *smaller* than the gaps between consecutive squares and powers of 2– enough so that the sum of their reciprocals diverges. Interesting!

Fermat Primality Test (base a) and Carmichael Numbers

The Little Fermat Theorem states that if *p* is prime and *a* is any integer *relatively prime to p* (i.e. *a* and *p* share no common factors) then $a^{p-1} \equiv 1 \bmod p$ (i.e. p divides $a^{p-1} - 1$), or equivalently, $a^p \equiv a \bmod p$. In other words, *p* divided into a^p has a remainder *a*. For example, 7 is prime and $2^7 = 128 = 7 \times 18 + 2$ or $2^7 = 2 \bmod 7$. Observe that 9 is not prime and 2 and 9 are relatively prime so $2^9 = 512 = 9 \times 56 + 8$ or $2^9 \neq 2 \bmod 9$.

The *converse* to the Little Fermat Theorem states if $a^{p-1} \equiv 1 \bmod p$ or equivalently, $a^p \equiv a \bmod p$ and *a is relatively prime to p* (which means *a* and *p* share no common factors) then *p* is prime. This might be a good test for *primality*. For example, if *p* is any odd integer (*p* and 2 are relatively prime) and if $2^p \equiv 2 \bmod p$ then *p* is prime – a possible test for primality?

Unfortunately, this converse to the Little Fermat Theorem, the *Fermat Primality Test*, is false.

Counterexample: $341 = 11 \times 31$ is the smallest *composite* number that demonstrates that the *Fermat Primality Test (base 2)* can give a *false result*; that is $2^{341} \equiv 2 \bmod 341$. That is 341 divided into 2^{341} (a very large number) has a remainder of 2 but 341 is composite, not prime. Hence 341 is called a *Fermat Pseudo-Prime* (sort of a *false positive* for primality), not a prime!

Aside: There are efficient mathematical techniques to perform calculations like $2^{341} \bmod 341$

However, had we used *3 instead of 2* we have $3^{341} \equiv 168 \bmod 341$ (341 divided into 3^{341} has remainder 168) so using the *Fermat Primality Test (base 3)* detects that 341 is not prime; 341 is not a *Fermat Pseudo Prime (base 3).*

But $3^{91} \equiv 3 \bmod 91$ makes 91 a *Fermat pseudo-prime base 3* except $91 = 7 \times 13$; 91 is not prime but composite. Of courses using 2 instead of 3 results in $2^{91} \equiv 37 \bmod 91$ so 91 is not a *Fermat pseudo-prime base 2* . Using 2 we would have immediately detected that 91 is not prime.

The Fermat Primality Test depends on the base *a* being used. One way to strengthen the Fermat Primality Testing would be to use multiple base *a's, for example a = 2, 3, 5, and 7 and if p passed the Fermat Primality Test for multiples bases, the odds are good (but not absolute) that p is indeed prime*. For large candidates *p*, this test is more efficient than sequentially dividing p by all primes less than $\sqrt{p}$

However – there is a problem!

There are composite numbers, called *Carmichael Numbers which always pass the Fermat Primality Test for all bases a.* That is, a *Carmichael Number p* has the property that $a^p \bmod p = a$ for all a such that $2 \leq a < p$ yet p is not prime. The first three Carmichael Numbers are $1105 = 5 \times 13 \times 17$, $561 = 3 \times 11 \times 17$, and $1729 = 7 \times 13 \times 19$ the famous taxicab number (see *1729*).

So what use are primes?

Answer: Public Key Encryption like the RSA algorithm. Factoring a large number is computationally very difficult. The RSA algorithm encryption uses a numeric key which is the product of *two very large primes*. It can only be broken if their product is factored – which is computationally difficult to do (although computers are getting faster and faster).

In outline the RSA Encryption Algorithm works like this

Alice wants to send a message to Bob. To do so she needs Bob's public key to encrypt the message.

Bob generates his public key as follows: Starting with two very large primes, *p* and *q*, he computes their product $n = p \times q$. Next Bob chooses a second integer e which is relatively prime to $(p-1)\cdot(q-1)$; that is e and $(p-1)(q-1)$ have no common factors. The numbers e and n are the *public* keys. So e and n are made known to Alice or to anyone else who wants to send an encrypted message to Bob.

The reason RSA Encryption is secure is that the key to breaking RSA requires factoring *n to recover p and q*. The primes *p* and q are the *private* keys known only to the *receiver Bob* who can use them to *decrypt* any *encrypted* message sent to him

In outline it works like this:

Encryption: Given a message M (a large integer), Alice using the public keys e and n encrypts M by computing $E = M^e \bmod n$. E is sent over an unsecure channel.

Decryption: To decrypt E and recover message M, Bob uses his private keys *p* and *q*. The mathematics is somewhat complex and among other results about primes, it makes use of Fermat's Little Theorem.

Mersenne Primes

Primes of the form $2^n - 1$ are called Mersenne Primes named after Fr. Marin Mersenne (1588 – 1648) a Minim Friar who corresponded with many of the French intellectuals and mathematicians of the early 17th century including Pascal, Descartes, and Roberval. For example, $2^2 - 1 = 3$, $2^3 - 1 = 7$, $2^5 - 1 = 31$ etc. are all examples of Mersenne Primes. In fact, if $2^n - 1$ is prime then so is *n*. Unfortunately, the converse is false; *n* can be prime but $2^n - 1$ is composite. For example, $2^{11} - 1 = 2047 = 23 \times 89$.

The GIMPS (Great Internet Mersenne Prime Search) is an organized search for Mersenne Primes using a volunteer network of computers. Currently the 47th Mersenne Prime (as of 8/2008) is $2^{43,112,609} - 1$. The largest known Mersenne Prime (as of 12/2018) is $2^{82,589,933} - 1$ which may be the 51st Mersenne Prime but unlike the 47th Mersenne Prime not all primes between 43,112,609 and 82,589,933 have been checked.

But There's More – The Euclid- Euler Theorem

Euclid (c. 300 BCE) proved that if $2^n - 1$ is prime then $\left(2^n - 1\right)2^{n-1}$ is a *perfect number*, that is a number whose proper divisors sum to that number. For example, $\left(2^2 - 1\right)2^{2-1} = 3 \cdot 2 = 6$ where the proper divisors of 6 sum to 6; that is $1+2+3=6$. Likewise, $\left(2^3 - 1\right)2^{3-1} = 7 \times 4 = 28 = 1+2+4+7+14$. Two thousand years later in 1747 Euler proved that every perfect number is in the form of $\left(2^n - 1\right)2^{n-1}$ where $2^n - 1$ is a Mersenne Prime. Thus, there is a tight connection between Mersenne Primes and even perfect numbers.

It is not known if there are odd perfect numbers.

"Without primes, though, the music would be boring,
the sing-song regularity, the constant whine
would drive all of us absolutely batshit bonkers."

Mirror Twin Primes, Palindromic Primes, Permutable (Absolute) Primes, repunit Primes

Given a prime, if you reverse the digits, is the result a prime? For example, 17 and 71 are *reversed digit primes* or *mirror primes* as are 13 and 31, 37 and 73, 79 and 97. 11 and 101 are *palindrome* primes (see below).

A *mirror twin prime* combines the properties of being both a *twin prime* and a *mirror prime* – for example the prime 73 is the twin of the prime 71 and is the mirror of the prime 37. 71 itself is also mirror twin prime since its mirror 17 is prime. The primes 13 and 31 are also *mirror twin primes* (11 is the twin of 13 and 29 is the twin of 31).

Now consider *linking* Mirror and Twin Primes:

37 - 73 - 71 - 17 – 19 – 91 = 7 × 13 or 11 = 13 = 31 = 29 - 92 = 2 × 2 × 23

A sequence of primes arranged so adjacent primes are either mirror twins or twin primes (i.e. the differ by 2) is a *mirror - twin prime* sequence.

Example: Seen above 19, 17, 71, 73, 37 is a *mirror – twin prime* sequence of length 5 (note that 91 is composite so 19 is only a twin prime while 37 is only a mirror prime as 35 and 39 are composite).

1303, 1301, 1031,1033, 3301, 3299, 9923 is a *mirror – twin prime* sequence of length 7.

Obviously mirror and palindrome primes begin and end with the digits 1, 3, 7, and 9;

A *palindromic prime* is a mirror twin prime with itself, like 11, 101, and 131. Scanning the digits forwards or backward is the same.

With a *permutable or absolute prime,* if you reorder the digits in any way, you have a prime. For example, 113, 131, and 311 are absolute primes as are 337, 373, 733, and 199, 919, 991.

A *repunit (repeated unit) prime* is a type of an *absolute prime* whose only digits are 1's – like 11. 1111111111111111111=R_{19} is the next largest repunit prime.

A List of Absolute Primes

2, 3, 5, 7, 11, 13, 17, 31, 37, 71, 73, 79, 97, 113, 131, 199, 311, 337, 373, 733, 919, 991, 1111111111111111111=R_{19}, 11111111111111111111111=R_{23}, R_{317}, R_{1031}

Note The notation Rn refers to a number consisting of n 1's -

A prime is a *mirror* prime, a *palindromic* prime or absolute prime *depending* on the radix or base used to represent the prime while being a twin prime is *independent of the base or radix* used to represent the prime. While 73, the twin of 71, is not a palindromic prime base 10, the *numeric value 73* is a palindromic prime base 2 since 73 = 1001001_2.

15_8 and 51_8 are *mirror* primes base 8 (decimal values 13 and 41) and both are twin primes base 8; that is 13_8 and 53_8 base 8 are decimal values 11 and 43 respectively.

Big Bang Theory, Sheldon, and the prime 73

In episode 73 of the Big Bang Theory Sheldon announces that 73 is the best number.

> 73 is the 21^{st} prime and its *mirror* twin 37 is the 12^{th} prime.
> From 37, the product of its two digits 3 and 7 is 21.
> 73 in binary is 1001001 a palindrome prime base 2.
> 17 the mirror twin of 71, the twin prime of 73 is also a binary palindrome: 10001_2

Note: Jim Parsons was born in 1973. In many shows he's wearing a t-shirt with the number 73 on it

"Thunderous lightning seldom strikes twice
in time and place. But sometimes

it does."

And Over and Again

Sieve of Eratosthenes

And over and again, tidal waves scrub, rinse, and sift cluttered sets: bones, jetsam, driftage of briny matter, myriad sea-gifts. Split bivalves, slime, starfish, nacreous scallops present bits to an opalescent ocean-deep of past living-treasure lairs. Primordial glittering stew washes in broad cycles, current traces plankton, drifting, shimmering towards the scrambled amalgam, anxious jumbled future. Splintered rift tears ruins of coral atolls, scours into mounds the smooth polished stone, roiling whirlpools' motes glistening into spindrift. Jostled, dancing, pattering sand leaves sounds in the shells; precious music, magic symphony, chattering echos, beautiful, watery abluvion, soft singing sea.

And tidal scrub and cluttered bones, driftage, briny myriad gifts, bivalve, starfish,
scallops' bits, an ocean of living lairs, glittering, washes broad current, plankton
shimmering. The amalgam, jumbled, splintered tears of atolls into the polished
roiling motes, into jostled pattering leaves. In shells, music symphony echoes
watery-soft sea.

– E.R. Lutken (3: A Taos Press © 2021)

Sieve of Eratosthenes

Eratosthenes (ca 284 – 192 BCE) was a famous 3rd century BCE Greek mathematician known among other things for his method for finding primes. His Sieve of Eratosthenes is a technique that is expressed poetically in the poem *Over and Over.*

Begin by listing out the numbers say from 2 to 100 (recall 1 is not considered a prime). Then starting at 2 cross out every 2nd entry. This eliminates all integers divisible by 2.For example …

	2	3	~~4~~	5	~~6~~	7	~~8~~	9	~~10~~
11	~~12~~	13	~~14~~	15	~~16~~	17	~~18~~	19	~~20~~
21	~~22~~	23	~~24~~	25	~~26~~	27	~~28~~	29	~~30~~
31	~~32~~	33	~~34~~	35	~~36~~	37	~~38~~	39	~~40~~
41	~~42~~	43	~~44~~	45	~~46~~	47	~~48~~	49	~~50~~
51	~~52~~	53	~~54~~	55	~~56~~	57	~~58~~	59	~~60~~
61	~~62~~	63	~~64~~	65	~~66~~	67	~~68~~	69	~~70~~
71	~~72~~	73	~~74~~	75	~~76~~	77	~~78~~	79	~~80~~
81	~~82~~	83	~~84~~	85	~~86~~	87	~~88~~	89	~~90~~
91	~~92~~	93	~~94~~	95	~~96~~	97	~~98~~	99	~~100~~

And over again, waves rinse, sift sets, jetsam of matter, sea-split slime, nacreous present to opalescent, deep past. Treasure, primordial stew in cycles, traces drifting toward scrambled, anxious future rift ruins. Coral scours mounds' smooth stone, whirlpools' glistening spindrift. Dancing sand sounds the precious, magic chattering, beautiful abluvion singing.

Rinse jetsam sea, nacreous, opalescent treasure in drifting anxious ruins, mounds, whirlpools dancing the chattering singing.

And over again: The next not-crossed-out number after 2 is 3. Starting at 3 cross out every 3rd entry (including those already crossed out).

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	25	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	35	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~
~~51~~	~~52~~	53	~~54~~	55	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	65	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	77	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	85	~~86~~	~~87~~	~~88~~	89	~~90~~
91	~~92~~	~~93~~	~~94~~	95	~~96~~	97	~~98~~	~~99~~	~~100~~

And over again, waves sift sets of matter, split slime, present to deep past. Primordial stew cycles, traces toward scrambled future rift; coral scours smooth stone, glistening spindrift. Sand sounds precious magic, beautiful abluvion.

Slime-deep, scrambled coral sounds beautiful.

And over again: After 3 the next-not-crossed-out number is 5. Cross out every 5th number starting at 5.

	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	49	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	77	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
91	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

And over again, waves sift sets of matter, split present to past. Primordial stew cycles, traces towards future rift, scours smooth stone, glistening spindrift, sand, precious, magic abluvion.

Traces glistening magic.

And over again: After 5 comes 7

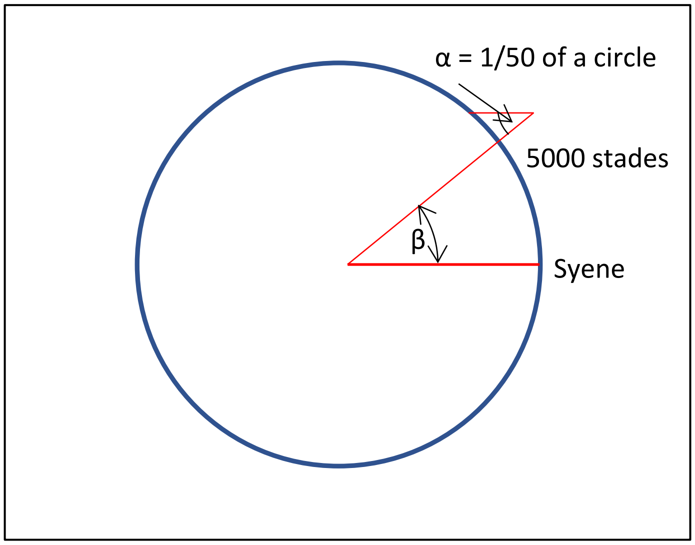
	2	3	~~4~~	5	~~6~~	7	~~8~~	~~9~~	~~10~~
11	~~12~~	13	~~14~~	~~15~~	~~16~~	17	~~18~~	19	~~20~~
~~21~~	~~22~~	23	~~24~~	~~25~~	~~26~~	~~27~~	~~28~~	29	~~30~~
31	~~32~~	~~33~~	~~34~~	~~35~~	~~36~~	37	~~38~~	~~39~~	~~40~~
41	~~42~~	43	~~44~~	~~45~~	~~46~~	47	~~48~~	~~49~~	~~50~~
~~51~~	~~52~~	53	~~54~~	~~55~~	~~56~~	~~57~~	~~58~~	59	~~60~~
61	~~62~~	~~63~~	~~64~~	~~65~~	~~66~~	67	~~68~~	~~69~~	~~70~~
71	~~72~~	73	~~74~~	~~75~~	~~76~~	~~77~~	~~78~~	79	~~80~~
~~81~~	~~82~~	83	~~84~~	~~85~~	~~86~~	~~87~~	~~88~~	89	~~90~~
~~91~~	~~92~~	~~93~~	~~94~~	~~95~~	~~96~~	97	~~98~~	~~99~~	~~100~~

And over again, waves sift sets of matter, split present to past. Primordial stew cycles toward future rift, scours smooth stone, spindrift, sand, precious abluvion.

- E.R. Lutken

And over again: After 7 is 11 but starting at 11 no more numbers are crossed out!–At this point all that remains (in black) are the primes between 2 and 100 – 25 of them to be exact.

Eratosthenes is also known for his very accurate mathematically based determination of the circumference of the earth.



Eratosthenes knew that at midsummer the sun was directly overhead in Syene, a town 5000 *stades* south of Alexandria. At the same time *in Alexandria* a small pole cast a shadow whose angle α with the pole was 1/50 the circumference of a circle. Since angle α equaled angle β, the arc subtended by angle β equaled 1/50 of the circumference of the earth so 50 × 5000 *stades* was the circumference of the earth. Assuming a *stade* was approximately 516.73 feet, this gave the earth's circumference to be 24,466 miles, a value remarkably close to the real value of 24,860 miles.

Triangular Numbers

"EYPHKA! Num = Δ + Δ + Δ"
- Carl Friedrich Gauss

One makes Two
Two Makes Three
Three makes all

naturals'
successive
summation

triangles
adding to
perfect squares

creator
preserver
destroyer

immortal
beings, earth
water, sky

mystical
three part soul
breath, wind, rest

three jewels
teacher, truth
collective

ancestors
moral code
transcendence

trinity
parent child
eidolon

messenger
sacred texts
lore, heart, word

-- E R Lutken (3: A Taos Press © 2021)

Carl Friedrich Gauss 1777 – 1855

1. Young Carl Friedrich Gauss

There is a story, perhaps apocryphal, about C.F. Gauss as a young boy. The story goes that Gauss's teacher asked the class to sum the integers from 1 to 100 figuring, I suppose, that this would take the students some time to complete. Young Carl quickly came up with the correct answer: 5050.

How did he do it?

We think that young Gauss realized that there were two ways to sum the integers from 1 to 100

– from low to high: $1+2+3+4+...+98+99+100$ and high to low: $100+99+98+...+3+2+1$.

Adding the two rows column by column

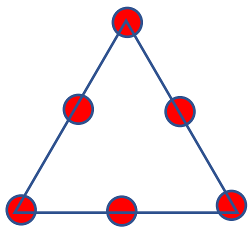
```
    1 +   2 +   3 + … +  98 +  99 + 100
+ 100 +  99 +  98 + … +   3 +   2 +   1
  -------------------------------------
  101 + 101 + 101 + … + 101 + 101 + 101 = 100×101 = 10100
```

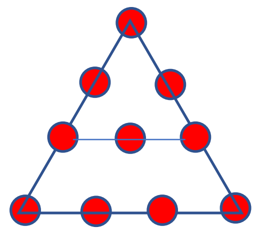
allowed him to easily compute twice the sum and thus compute the final answer: 5050!

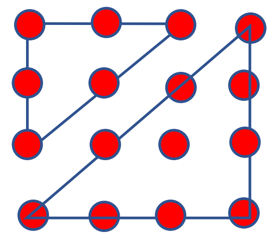
This is the 100th *triangular number*

2. *Triangular Numbers*

The n^{th} triangular number, denoted as T_n, is the sum of the integers from 1 to n. For example, the third triangular number, T_3, is 6 as 1 + 2 + 3 = 6 while the fourth triangular number, T_4, is 10 as 1 + 2 + 3 + 4 = 10. They are called triangular numbers because, as seen below, you can arrange 6 dots or 10 dots in a triangular configuration.







Adding 5 dots to the base of T_4 obtains T_5, the next sized triangle. Thus, the connection between the sum of the integers 1 thru *n* and the corresponding triangular configuration of dots is easily seen.

"naturals'
successive
summation"

Moreover, summing two adjacent triangular numbers, for example $T_3 + T_4$ gives you a square, in this case $6 + 10 = 16 = 4^2$.

"triangles
adding to
perfect squares"

Using young Gauss's trick-calculation we can easily obtain a *closed form* for the n^{th} triangular number by summing 1 through n both forwards and backwards, adding the two rows column by column, multiplying each "n+1" sum by n (since we have n sums) and dividing by 2.

```
      1 +   2 +   3 + … + n-2 + n-1 + n
 +    n + n-1 + n-2 + … +   3 +   2 + 1
   -------------------------------------
   n+1 + n+1 + n+1 + … + n+1 + n+1 + n+1 = n×(n+1)  = n^2+n
```

Result: $T_n = 1 + 2 + 3 + \ldots + (n-2) + (n-1) + n = \frac{n^2 + n}{2} = \frac{n(n+1)}{2}$

Using this formula, one can easily prove the sum of T_{n-1} and T_n equals n^2.

Result: $T_{n-1} + T_n = \frac{(n-1)n}{2} + \frac{n(n+1)}{2} = \frac{n^2 - n + n^2 + n}{2} = \frac{2n^2}{2} = n^2$

3. *Gauss' Eureka Theorem*

While it was well known that every square is the sum of two triangular numbers, Gauss showed that every integer is the sum of at most *three* triangular numbers - or allowing 0 to be the 0^{th} triangular number, every integer is the sum of three triangular numbers. Hence:

"EYPHKA! Num = Δ + Δ + Δ"
- Carl Friedrich Gauss

This is Gauss's "Eureka Theorem" dated July 10, 1796 – when Gauss was 19 years old, the 18^{th} entry in his diary which read "EYPHKA! num = Δ + Δ + Δ". The proof is rather complicated but easy to demonstrate. First, it's easy to list out triangular numbers:

0, 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, 105, …

Then given any integer *n*, it's simply a matter of finding three triangular numbers (see above) to sum to *n.*

Example: 17 = 10 + 6 + 1 = 15 + 1 + 1
35 = 28 + 6 + 1 = 15 + 10 + 10
100 = 91 + 6 + 3 = 78 + 21 + 1

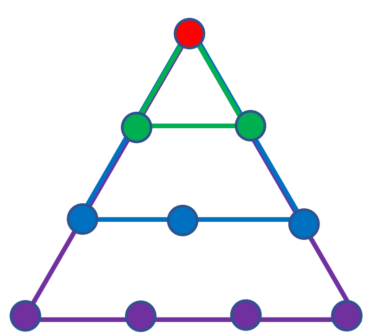
There may be more than one way to decompose an integer *n* into a sum of three triangular numbers.

A Footnote

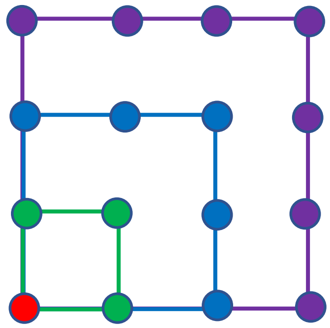
The arrangement of the poem is the 4^{th} triangular number 10: an arrangement also known as the **tetractys** which had some significance to the Pythagoreans, early mathematical mystics who coined the phrase "*All is Number*" – a statement that somehow mathematics is the key to understanding the natural world around us.

Beyond Triangular and Square Numbers – Figurate Numbers

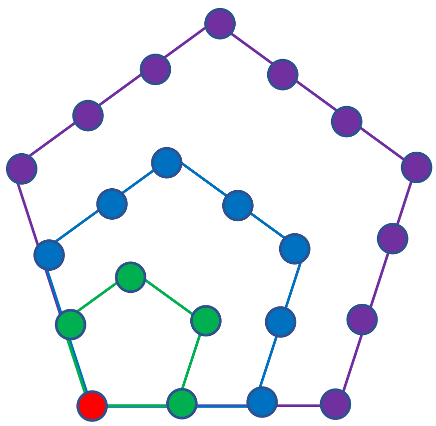
The n^{th} triangular number T_n is the sum of the first n integers: $T_n = 1+2+3+...+n$ as shown above. The *closed form* equation for the nth triangular number is given by $T_n = \frac{n \cdot (n+1)}{2}$



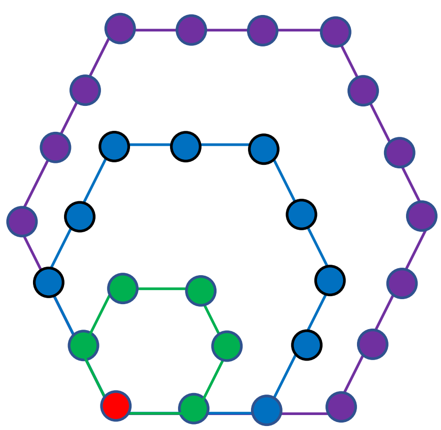
The nth square number n^2 is the sum of the first n odd integers: $n^2 = 1+3+5+...+2n-1$ as demonstrated below. Note that the difference of two adjacent squares is an odd integer and that every odd integer is the difference of two adjacent squares.



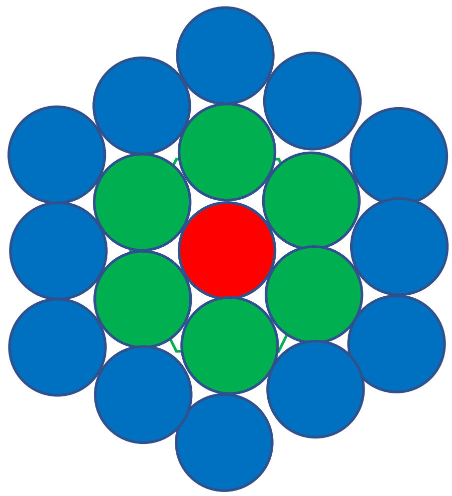
But why stop here? There are pentagonal numbers where $P_n = 1+4+7+...+3n-2$ (note the increase by three – see below). The *closed form* equation is $P_n = \frac{3n^2 - n}{2}$



There are *cornered* hexagonal numbers $H_n = 1+5+9+...+4n-3$ (note the increase by four) whose closed form equation is $H_n = 2n^2 - n$ etc …



There are so *centered* hexagonal numbers where $CH_n = 1 + 6 + 12 + ... + (n-1)\cdot 6$ whose close form equation is $CH_n = 3n^2 - 3n + 1$



And so on ….

And finally, the *Fermat Polygonal Number Theorem* which states that every integer can be expressed as the sum of n *n-gonal* numbers (using 0 is the zeroth n-gonal number) where Gauss' Eureka Theorem is a special case. For example …

Linear numbers (aka the natural numbers) : 1, 2, 3, 4, 5, …

Triangular numbers: 1, 3, 6, 10, 15, 21, 28, 36, 45, 55, 66, 78, 91, … 100 = 91 + 6 + 3

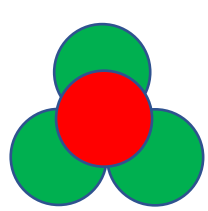
Squares: 1, 4, 9, 16, 25, 36, 49, 64, 81, 100, … 99 = 64 + 25 + 9 + 1

Pentagonal Numbers: 1 5, 12, 22, 35, 51,70, 92, 117, … 100 = 92 + 5 + 1 + 1 + 1

Back to Triangular and Square Numbers: Pyramidal Numbers

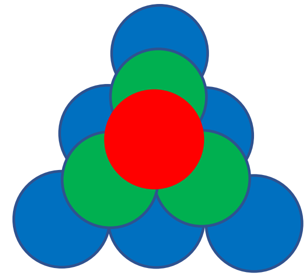
Another feature of triangular (and square) numbers is that they appear in the stacking of a set of round objects, like cannonballs.

One cannonball can be stacked atop three cannonballs as follows.



And in turn three cannonballs can be stacked atop six cannonballs with one cannonball stacked atop the three.

And the six cannonballs can be stacked atop ten and ten atop fifteen etc.

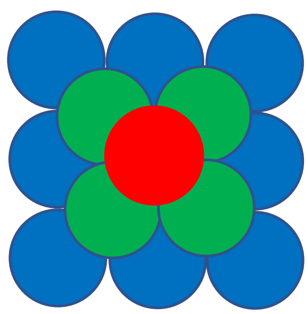


This yields the sequence **1**, 1+3 = **4**, 1 + 3 + 6 = **10**, 1 + 3 + 6 + 10 = **20** …where the *sum of the first n triangular numbers* is the number triangular stacked cannonballs in a pile of height *n*. In general, this is …

$$S_t(n) = \frac{n^3 + 3n^2 + 2n}{6} = \frac{n(n+1)(n+2)}{6}$$

The same can be done with *square* stacking of cannonballs: 1 on top of 4 on top of 9 on top of 16 etc. yielding the sequence **1**, 1 + 4 = **5**, 1 + 4 + 9 = **14**, 1 + 4 + 9 + 16 = **30** … where the sum of the first n squares is the number of square stacked cannonballs in a stack of height n. In general, this is …

$$S_s(n) = \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n+1)(n+1)}{6}$$



Sums of Reciprocals?

Finally, what about the *sum of the reciprocals of the triangular numbers*:

$$\sum_{k=1}^{\infty} \frac{1}{T(k)} = 1 + \frac{1}{3} + \frac{1}{6} + \frac{1}{10} + \frac{1}{15} + \dots$$

Leibniz in Paris 1672-1676

It was while Gottfried Wilhelm Leibniz (1646-1716), the co-discovered/co-inventor of calculus with Sir Issac Newton, was in Paris on a diplomatic mission that Leibniz learned mathematics. Knowing nothing or little about mathematics he sought help from the Dutch mathematician Christian Huygens(1629-1695) who assigned this problem to Leibniz to see if he, Leibniz, could solve the problem of computing the sum of the reciprocals of the triangular numbers; that is $1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\ldots+\frac{2}{n(n+1)}+\ldots$, an infinite series which (unlike the Harmonic Series) converges.

Here is the solution:

Let $S=1+\frac{1}{3}+\frac{1}{6}+\frac{1}{10}+\frac{1}{15}+\ldots$

Then $\frac{1}{2}S=\frac{1}{2}+\frac{1}{6}+\frac{1}{12}+\frac{1}{20}+\frac{1}{30}+\ldots$

which can be expressed as $\frac{1}{2}S=\frac{1}{2}+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\left(\frac{1}{4}-\frac{1}{5}\right)+\left(\frac{1}{5}-\frac{1}{6}\right)+\ldots$

which with some regrouping yields

$$\frac{1}{2}S=\left(\frac{1}{2}+\frac{1}{2}\right)+\left(-\frac{1}{3}+\frac{1}{3}\right)+\left(-\frac{1}{4}+\frac{1}{4}\right)+\left(-\frac{1}{5}+\frac{1}{5}\right)+\left(-\frac{1}{6}+\frac{1}{6}\right)+\ldots$$

But everything except the first two terms cancels leaving $\frac{1}{2}S=\frac{1}{2}+\frac{1}{2}$

Therefore if $\frac{1}{2}S=1$ then $S=2$ so $\sum_{k=1}^{\infty}\frac{1}{T(n)}=2$

And the rest (for Leibniz and mathematics) is history.

Recall that Leonard Euler solved the famous *Basel Problem* where he computed the sum of the reciprocals of the squares (recall π)

Pythagorean Triples

A *Pythagorean Triple* is a triple of three integers a, b, and c such that $a^2+b^2=c^2$, for example 3,4, and 5 is a Pythagorean Triple since $3^2+4^2=9+16=25=5^2$

Since every odd integer is the difference between two adjacent squares, there is a method to find a Pythagorean Triple given that the smallest component is an *odd* integer. Then the other two components can be obtained from the first since the *square of an odd integer is also odd*.

For example, $9=3^2$ is the odd square of an odd integer. It follows that $16=4^2=\left(\frac{3^2-1}{2}\right)^2$ and $25=5^2=\left(\frac{3^2+1}{2}\right)^2$. Therefore, using as little algebra …

$$3^2+4^2=3^2+\left(\frac{3^2-1}{2}\right)^2=\frac{4\cdot 3^2+3^4-2\cdot 3^2+1}{4}=\frac{3^4+2\cdot 3^2+1}{4}=\left(\frac{3^2+1}{2}\right)^2=25$$

This method of obtaining the other two Pythagorean triple members from the first generalizes.

Result: If *m* is an odd integer greater than 1, then

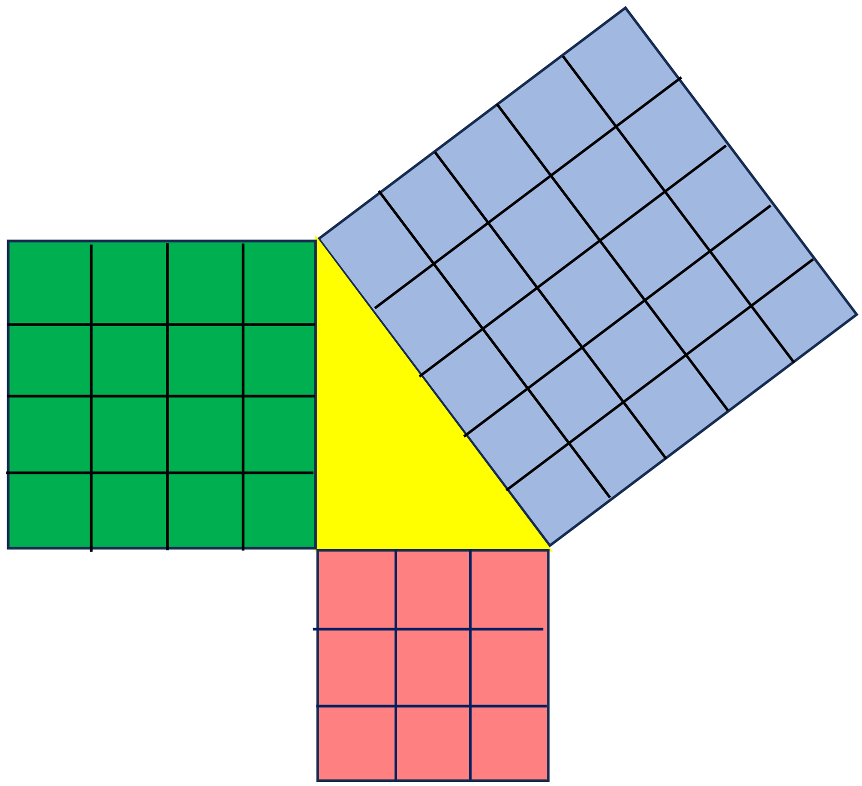
$$m^2+\left(\frac{m^2-1}{2}\right)^2=\left(\frac{m^2+1}{2}\right)^2 \text{ and } \left(m,\quad \frac{m^2-1}{2},\quad \frac{m^2+1}{2}\right) \text{is a Pythagorean Triple}$$

Note that if *m* is odd and greater than 1 both $\frac{m-1}{2}$ and $\frac{m+1}{2}$ are integers, and

$$m^2+\left(\frac{m^2-1}{2}\right)^2=\frac{4m^2}{4}+\frac{m^4-2m^2+1}{4}=\frac{m^4+2m+1}{4}=\left(\frac{m^2+1}{2}\right)^2$$

Therefore, the result follows.

Example: 7, $\left(\frac{7^2-1}{2}\right)=24\,,25=\left(\frac{7^2+1}{2}\right)$ yields $7^2+24^2=49+576=625=25^2$



If we order Pythagorean Triples from smallest to largest elements, it then follows that every odd integer is (can be) the smallest element of a Pythagorean Triple. There are Pythagorean triples of all even integers, for example 6,8,10, but no power of 2 can be smallest element of a Pythagorean Triple.

Euler's Identity

$$e^{i\pi} + 1 = 0$$

Ripples from tossed stones, snow-moon's ring,
gray eyes' irides, gold leopard's bane,
sun's bright halo, heavy crowns of kings,
kurgans of Scythians scattered across cold plains.
From one around to one, each circle owns
a ghostly heart within a complex plane,
forged like Achilles' burnished shield,
destroyed with flashing tracks of weapons drawn
in clash of purpose. Blood-soaked battlefields
preserve sublime geometry unrevealed.
Real lives never curl back into the womb.

First quarter moon, rainbows, mountain domes,
courses of sun, stars across the sky,
spear's arc traveling towards crimson home,
lids shadowing an archer's piercing eyes.
Half circles rise and fall from leaden ground
to ground, true axis for the maps of lives.
Wild Xanthos and Balios, under iron arm
compelled to follow radians in a turn,
pull Hector's broken body past the new-filled urn.
The root of death always comes too soon;
e to the i π equals negative one.

-- E R Lutken (3: A Taos Press © 2021)

"e to the i π equals negative one."

$$e^{i\pi} = -1$$

$$e^{i\pi} + 1 = 0$$

Revisiting the constant e

In *Phaeton's Ride* the constant *e* was introduced as the sum of the infinite series

$$e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{1}{k!} \approx 2.718281828\ldots$$

The function $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots = \sum_{k=0}^{\infty} \frac{x^k}{k!}$ was introduced as the *exponential function* which increased rapidly toward $+\infty$ as $x \to \infty$, that is $\lim_{x\to\infty} e^x = +\infty$ while e^x approached 0 asymptotically as $x \to -\infty$, that is $\lim_{x\to-\infty} e^x = 0$. And of course, e^x equals 1 for $x = 0$. Thus, exponential growth and decay are modeled by the function $f(x) = e^x$.

However, it is also interesting that the constant *e* has a close relationship with four other famous constants in mathematics, as expressed in Euler's Identity.

Euler's Identity - The Most Beautiful Equation in Mathematics

Euler's Identity, $e^{i\pi}+1=0$ combines in one equation five important numbers in mathematics:

$e \approx 2.718281828...$, the base of the exponential function e^x

i, the square root of minus 1: $i=\sqrt{-1}$

π, the ratio of the circumference of a circle to its diameter: $\pi = \dfrac{circumference}{diameter}$

1, the multiplicative identity: $a \times 1 = 1 \times a = a$

0, the additive identity: $a+0=0+a=a$

Demonstrating (Proving) the Connection

Recall that function e^x has an infinite series representation …

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^k}{k!} + ...$$

which if you differentiate it (term by term) yields back e^x (recall $\frac{d}{dx}\left(x^n\right) = n \cdot x^{n-1}$ for $n>0$)

$$\frac{d}{dx}\left(e^x\right) = \frac{d}{dx}\left(\sum_{k=0}^{\infty} \frac{x^k}{k!}\right) = \frac{d}{dx}\left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^k}{k!} + ...\right) = 0 + 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + ... + \frac{x^k}{k!} + ... = e^x$$

That is, e^x is the unique function which is *its own derivative*! Because of this unique property, among others, e^x is a very important and useful function!

Now when we evaluate e^x at $i \cdot x$ where $i=\sqrt{-1}$

$$e^{ix} = \sum_{k=0}^{\infty} \frac{(ix)^k}{k!} = 1 + i \cdot x - \frac{x^2}{2!} - \frac{i \cdot x^3}{3!} + \frac{x^4}{4!} + ... + \frac{(i \cdot x)^k}{k!} + ...$$

the infinite series can be partitioned into two series, one with real terms (even powers of x), the other with imaginary terms (odd powers of x).

$$= \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + (-1)^k \frac{x^{2k}}{(2k)!} + ..$$

$$+ \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!} i = x \cdot i - \frac{x^3}{3!} \cdot i + \frac{x^5}{5!} \cdot i - \frac{x^7}{7!} \cdot i + ... + (-1)^k \frac{x^{2k+1}}{(2k+1)!} i + ...$$

Now like e^x, the trig function sin(x) and cos(x) also have power series expansions (called Taylor Series expansions); that is

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + ... + (-1)^k \frac{x^{2k+1}}{(2k+1)!} + ..$$

and

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + ... + (-1)^k \frac{x^{2k}}{(2k)!} + ..$$

But looking back at the real and imaginary parts of the series expansion for e^{ix} given above, the real part is the same as the series for $\cos(x) = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k}}{(2k)!}$ and the imaginary part is i times the series for $i \cdot \sin(x) = i \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{(2k+1)!}$. In other words …

$$e^{ix} = \cos(x) + i \cdot \sin(x)$$

So the *trig functions* $\sin(x)$ and $\cos(x)$ are connected to the *exponential function* e^{ix}, all three of which can be expressed as *polynomial-like infinite series* – a result due to Leonard Euler (1707 – 1783).

Since $\cos(\pi) = -1$ and $\sin(\pi) = 0$ it follows that $e^{i\pi} = \cos(\pi) + i\sin(\pi) = 1 + i \cdot 0 = -1$. Therefore

$$e^{i\pi} + 1 = 0$$

$i = \sqrt{-1}$, $e^{i\pi}$, and the Geometry of Complex Numbers

Recall that a *complex number* is of the form $a + bi$ where *a* and *b* are real numbers, for example $3 + 2i$. It wasn't until the 17th and 18th centuries that mathematicians accepted complex numbers as being *numbers;* that they obeyed all the algebraic laws of numbers (or as I like to say they "*played well with other numbers*"). So if you add, subtract, multiply, divide, exponentiate or take a root of a complex number, the result is a complex number: they are *algebraically closed* . (Recall that $\sqrt{-1}$ is not *called* a "*real*" number although to call it *imaginary* is somewhat of a misnomer).

Addition and *subtraction* of complex numbers is straightforward: you add or subtract the real parts and the imaginary parts separately: $(3 + 2i) + (-1 + i) = (2 + 3i)$.

Multiplication uses the FOIL (First, Outside, Inside, Last) method that most of us learned in high school algebra. $(3 + 2i) \times (-1 + i) = -3 + 3i + -2i + 2i^2 = -5 + i$ (Remember $i^2 = -1$.)

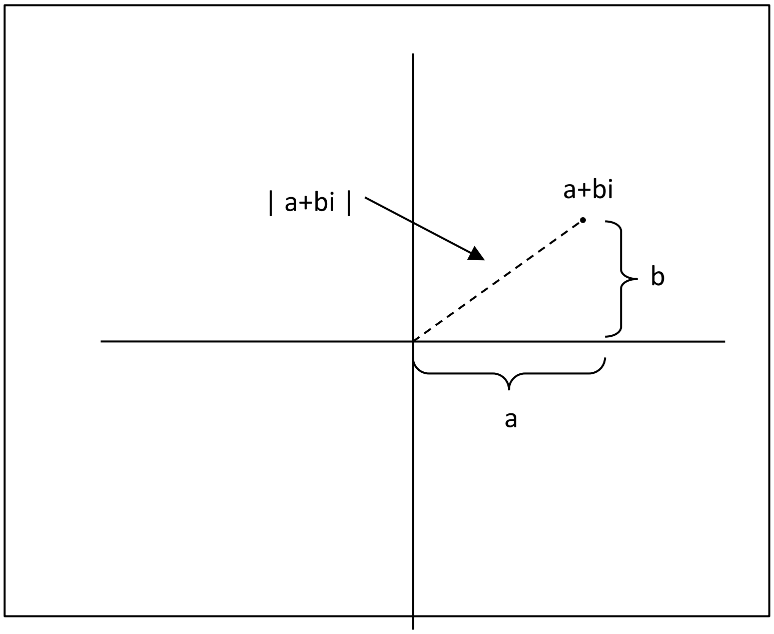
Unfortunately, unlike real numbers, complex numbers are not *well ordered*. That is given two complex numbers $3 + 2i$ and $-1 + i$ neither $3 + 2i \geq -1 + i$ nor $3 + 2i \leq -1 + i$ make sense. However if you multiply any complex number $a + bi$ by its *complex conjugate* $a - bi$ you get the positive sum $a^2 + b^2$ We define the *magnitude* of a complex number $|a + bi| = \sqrt{a^2 + b^2}$ so we can in a sense *order complex number by their magnitude although two different complex numbers can have the same magnitude.*

Division of complex numbers is done my multiplying numerator and denominator by the *complex conjugate of the denominator*. So to divide *3+2i* by *-1+i* …

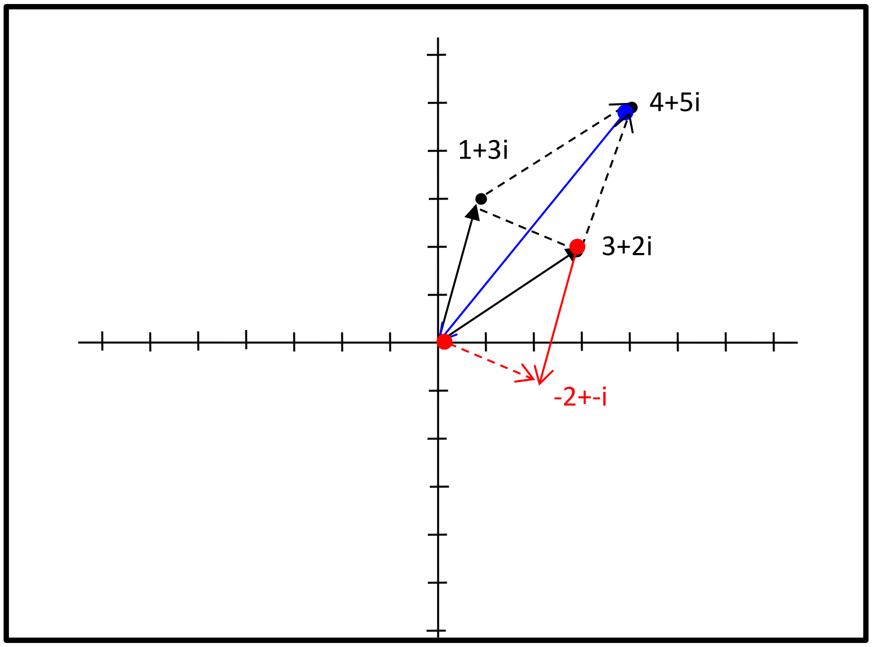
$$\frac{3+2i}{-1+i} = \left(\frac{3+2i}{-1+i}\right)\left(\frac{-1-i}{-1-i}\right) = \frac{-1-5i}{2}$$

The Complex Plane

Just as the real number line is used to represent real numbers, a *two-dimensional complex plane* can be used to represent complex numbers where the imaginary axis is erected at right angles to the real axis. There is a natural *one to one correspondence* between points in the complex plane and complex numbers of the form $a+bi$ where a is the distance along the horizontal real axis and b is the distance along the vertical imaginary axis. Since magnitude $|a+bi|$ equals $\sqrt{a^2+b^2}$, by the Pythagorean distance formula, the magnitude of $a+bi$, $|a+bi|$ is the *distance* from the origin to the point $a+bi$.



The Geometry of Addition & Subtraction – Parallelogram Rules



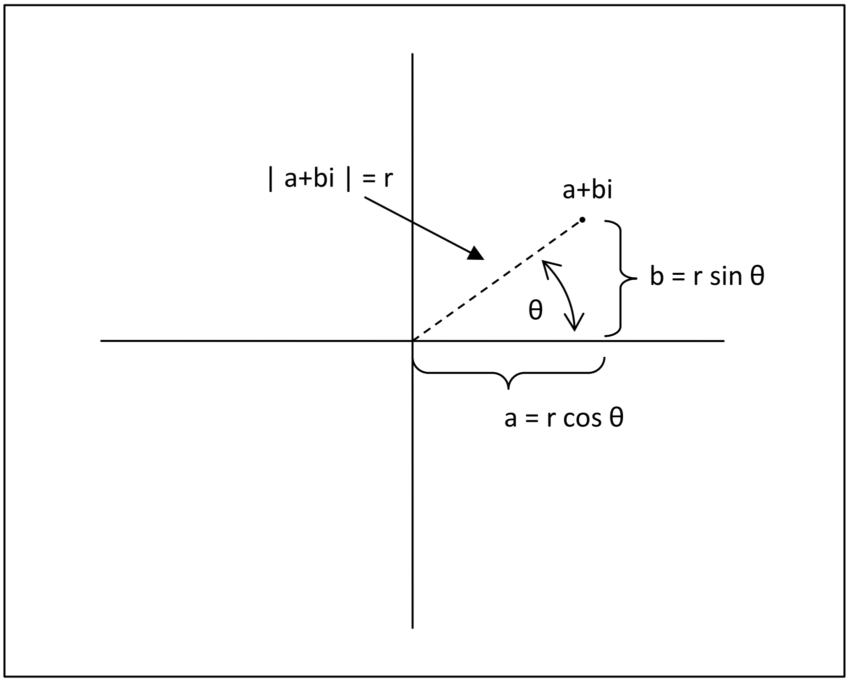
There is a simple geometric description of addition and subtraction. If you treat a complex number $a+bi$ like a vector (or arrow) whose tail is the at the origin and whose head is at the point $a+bi$ (so the magnitude is the length of arrow), then the addition of two complex numbers can be seen as placing the tail of one vector at the head of the other with the resulting arrow being the sum.

For example (3+2i) + (1+3i) = 4+5i

As you can see, the two arrows form two sides of a parallelogram – hence the parallelogram laws.

To subtract *reverse* the subtractor (subtrahend) arrow and place its head at the head of the minuend arrow. That is (3+2i) – (1+3i) = -2 - i. The resulting difference arrow is parallel to the other diagonal of the *addition* parallelogram.

The Polar Form for Complex Numbers: $z = a+bi = r\cdot(\cos\theta + i\sin\theta) = r\cdot e^{i\theta}$



If θ is the angle between the ray from the origin to the point $z = a+bi$ and the positive real axis, then right triangle trigonometry shows that $a = r\cos(\theta)$ and $b = r\sin(\theta)$. Putting this together obtains

$$a+bi =$$
$$r\cdot\cos\theta + r\cdot\sin\theta\cdot i =$$
$$r\cdot e^{i\theta}$$

where

$r = |z| = |a+bi| = \sqrt{a^2+b^2}$ is the *magnitude* or *modulus* of z and θ is called the *angle* or *argument* of z. This is the *polar form* of a complex number.

That is to say θ is the angle whose tangent is $\frac{b}{a}$ or $\theta = \tan^{-1}\left(\frac{b}{a}\right)$. Thus, z can be expressed as

$$z = a + bi = r \cdot \cos\theta + r \cdot \sin\theta \cdot i = r \cdot (\cos\theta + i\sin\theta) = r \cdot e^{i\theta}.$$

Using our knowledge of sines and cosines for certain angles we can convert between rectangular and polar representations. For example

$2 + 2i = \sqrt{8}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \sqrt{8} \cdot e^{i\cdot\frac{\pi}{4}}$ where the argument (angle) of $2 + 2i$ is $\frac{\pi}{4}$ and the magnitude is $2\sqrt{2}$.

$1 + \sqrt{3}i = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right) = 2 \cdot e^{i\cdot\frac{\pi}{3}}$ where the argument (angle) of $1 + \sqrt{3}i$ is $\frac{\pi}{3}$ and the magnitude is 2

Using the polar form for complex numbers multiplication is $\left(\sqrt{8} \cdot e^{i\cdot\frac{\pi}{4}}\right) \cdot \left(2 \cdot e^{i\cdot\frac{\pi}{3}}\right) = 2\sqrt{8} \cdot e^{i\cdot\frac{7\pi}{12}}$; that is geometrically speaking, multiplication of two complex numbers is *multiplying their magnitudes* and *rotating by the sum of their angles*. In particular, $e^{i\cdot\frac{7\pi}{12}}$ is a rotation of the unit vector counter-clockwise by an angle of $\frac{7}{12}\pi$ radians (105 degrees) .

Thus $e^{i\pi}$ is a rotation of the unit vector by π radians or 180 degrees ending up at -1. So

$$e^{i\cdot\pi} + 1 = 0$$

This geometry of complex numbers makes for some interesting effects as will be seen in *Augury in Sand.*

1,729

From faintly oscillating background fields
of ancient tides rocking through generations,
emerged a wave of unpremeditated energy
welling from crowds flooding in sacred migrations,
past scattered promontories in uneasy seas.

Rare genius, a limiting case of the breather solution
rose up in a flow of exquisite intensity,
eventually noticed by the West at a quick turn
then mad dash to comprehend the mystical singularity,
Peregrine Soliton of one sweeping mind.

Oceans of numbers rippling over this brain
aware of the flash at the surface of each dimpled wave:
seven times thirteen times nineteen,
smallest sum of 2 positive cubes in 2 different ways,
1729, a glint gracing the seascape vision of Ramanujan.

-- E R Lutken (3: A Taos Press © 2021)

Ramanujan - December 22, 1887- April 26, 1920)

Srinivasa Aiyangar Ramanujan was perhaps India's most famous mathematician. With very limited formal mathematical education he made substantial contributions to the analytical theory of numbers working on elliptic functions, continued fractions, and infinite series. He came to the attention of the well-known mathematician G.H. Hardy (Cambridge University) who arranged for Ramanujan to be brought to Cambridge in 1914 thus beginning an extraordinary though difficult collaboration as Ramanujan was plagued by poor health and ill-suited to the colder English climate. While mathematically brilliant, Ramanujan had difficulties due to his lack of formal mathematical education. Having key mathematical insights is not enough, one must provide a rigorous proof.

Ramanujan returned in India in 1919 where due to his poor health he died the following year at age 33 – the tragedy of genius.

1729 – The Taxi-Cab Number

The story of the 1729 as the Taxicab numbers as told by Hardy goes like this:

"I remember once going to see him [Ramanujan] when he was lying ill at Putney. I had ridden in taxi-cab No. 1729, and remarked that the number seemed to be rather a dull one, and that I hoped it was not an unfavourable omen. "No," he replied, "it is a very interesting number; it is the smallest number expressible as the sum of two [positive] cubes in two different ways."

$1729 = 7 \times 13 \times 19$ - 1729 is composite (not prime).
1729 is a *Carmichael Number* (see *Prime Number Song*)
1729 is the smallest number which is the sum of two cubes two different ways.

$$1729 = 9^3 + 10^3 = 729 + 1000$$
$$1729 = 1^3 + 12^3 = 1 + 1728$$

"seven times thirteen times nineteen,
smallest sum of 2 positive cubes in 2 different ways,
1729, a glint gracing the seascape vision of Ramanujan"

Ramanujan and pi

In 1914 Ramanujan produced the following remarkable equation for 1/π.

$$\frac{1}{\pi} = \frac{\sqrt{8}}{9801} \sum_{n=0}^{\infty} \frac{(4n)!}{(n!)^4} \cdot \frac{[1103 + 26390n]}{396^{4n}}$$

Ramanujan's formula is *truly remarkable* in that unlike previous formulas used to determine pi as presented in the previous essay on π which were obtained using geometric properties of circles (Archimedes), or the integration of the area under a semi-circle (Newton) or power series expansions of inverse trigonometric functions (Gregory, Leibniz, Nilakantha), there seems to be no obvious connection between Ramanujan's formula and the ratio of the circumference of a circle divided by its diameter.

Computing just the *first three terms* from the above equation yields 1/π accurate to 16 digits (and π to 15 digits); that is for n = 2

$$\frac{1}{\pi} = 0.3183098861837907 \text{ and } \pi = 3.141592653589793$$ [3]

Each additional term adds 8 more digits of accuracy to 1/π. As mentioned in the previous discussion of π, the arctangent-based equations used to determine π are *ultimately* too slow. Ramanujan's work opened up a new approach to the determination of π based on faster converging recursive formulas.

Ramanujan left notebooks containing formulas and equations which today are still being studied by mathematicians in attempts to understand and to provide proofs for his work.

"Peregrine Soliton of one sweeping mind"

[3] The values for 1/π and π were obtained using a Python program where floating point numbers (32 bits) are restricted to 16 decimal digits.

Augury in Sand

To those poor souls who dwell in night…
– William Blake

I came halfway around the world
to this beach of ash-gray basalt grains,
Hawai'i, surrounded by an infinity
of ocean, after I got news second-hand
she'd died. Seemed like an eternity
on the airplane, five stagnant hours.

Scratched at a crossword for two hours,
gave up, stared at the bloated world
of clouds, cheap window on eternity.
Ideas of an afterlife go against the grain
for me, too much dissonance at hand
to imagine angel-winged infinity.

Fractals were our chosen drafts of infinity,
Mandelbrot and Julia sets we'd peruse for hours,
watching mesmeric spokes form under hand:
seahorse tails, double spirals, island worlds,
crowns shrinking smaller than dust grains,
wild acid trips we mapped into eternity.

A slippery concept, Eternity
and her sister in space, Infinity,
positioning us on their respective lines, grains
in the middle of Here, at the Now hour,
where I pace empty shores of a dull world.
Before she died, we had everything in hand,

but on this beach, dreams drip from my hands.
Why should this be the now of eternity?
Why not some other speck of the moving world?
Plenty of room, stretching to infinity.
Pick a place and an hour, any other hour,
where sets of memories scatter like grain.

No grand recursive frills are worth one grain.
Nothing rewrites the flat, brutal history on hand.
I watch waves break hour to hour to hour,
iterations lapping at the lacework of eternity,
washing away all traces of diagrammed infinity,
while her simple, endless absence is seared into my world.

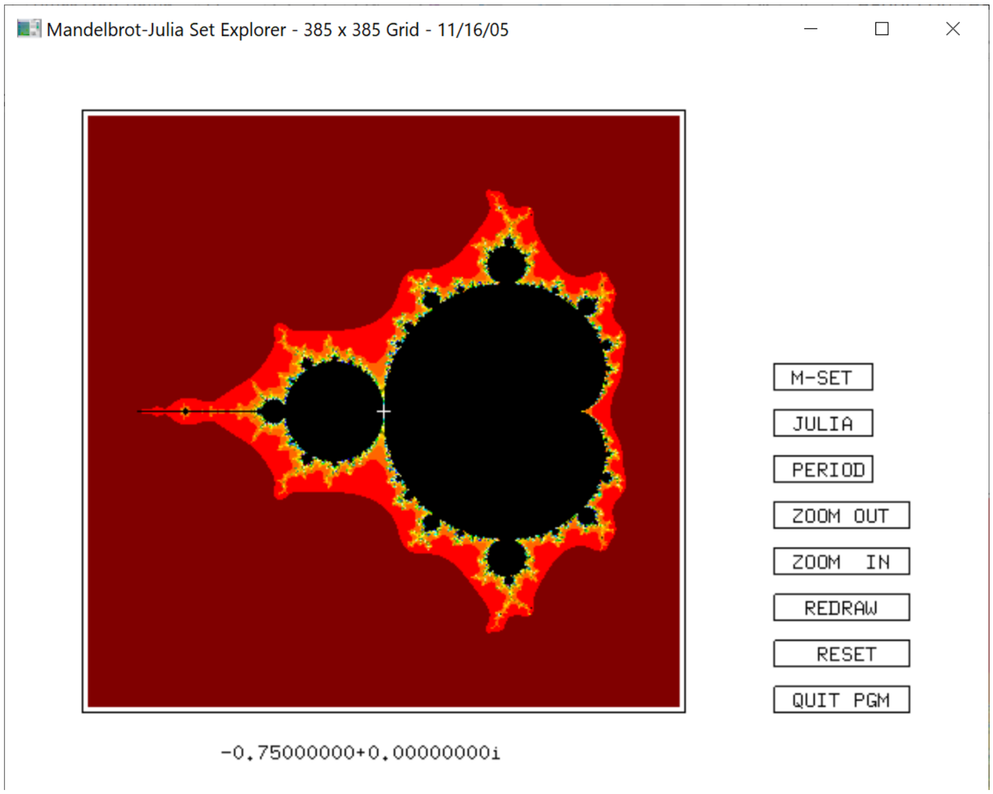
Eternity crawls by every hour,
infinity strikes hard at hand.
The world is nothing but a sand grain.

--E R Lutken (3: A Taos Press © 2021)

Mandelbrot and Julia Sets

"Fractal were our chosen drafts of infinity,
Mandelbrot and Julia set we'd peruse for hours …"

Endless hours can be spent using a computer exploring Mandelbrot and Julia sets – perhaps one of most complicated mathematical objects known.



The Mandelbrot Set lives in the complex plane – generated by the simple recursive equation $z_{n+1} = z_n^2 + c$ where z and c are complex numbers

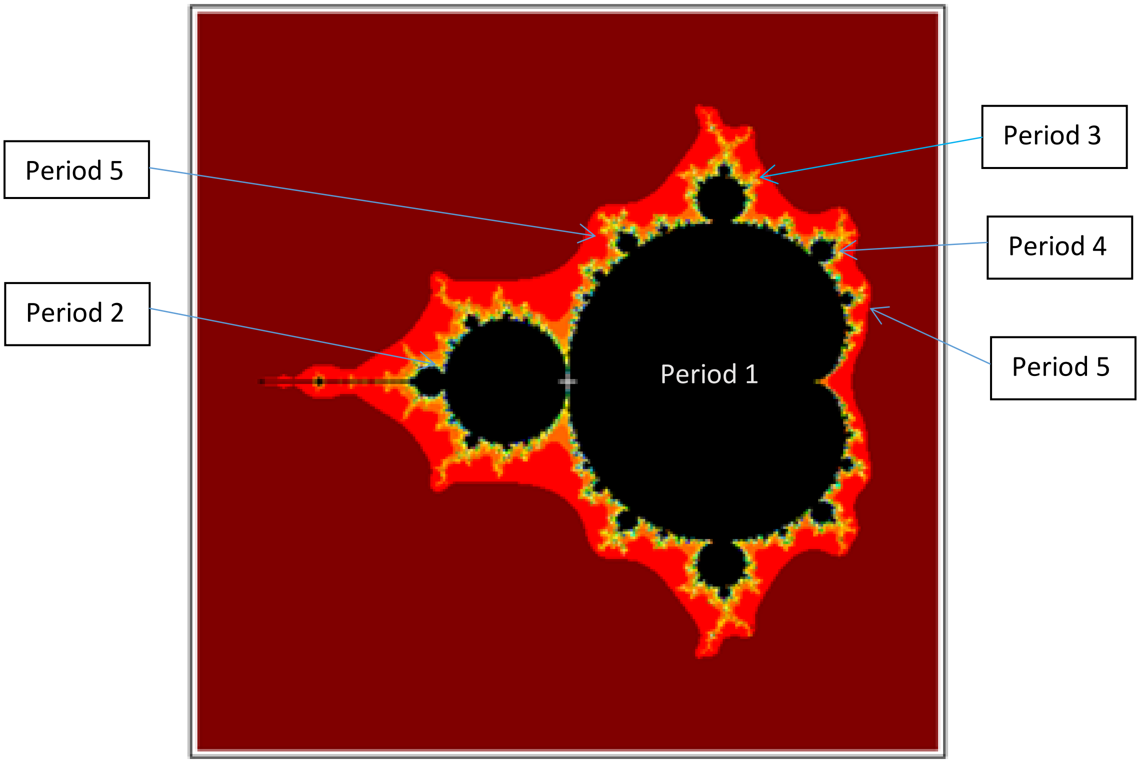
Fix *c* and iterate $z_{n+1} = z_n^2 + c$ starting with $z_0 = 0 + 0i$ If z_n does not escape to infinity, then *c* (in the *c* complex plane) is in the Mandelbrot set. Color that *point* black. It's helpful at this point to recall the geometry of complex numbers and in particular, the effect of multiplication of complex numbers (see *Euler's Identity*). If you *multiply* two complex numbers, say $a \times b$, the effect is to stretch *a* (the arrow vector from the origin to the point *a* in the complex plane) by the magnitude of vector *b* and to rotate the resulting vector around the origin (by the sum of the two angles the vectors make with the positive real axis) . This can make for some very interesting results which is why the Mandelbrot and Julia sets, when graphed, are so interesting.

Generating the Mandelbrot Set

The above picture of the Mandelbrot Set was generated using an *escape algorithm*. Pick a complex value *c* in the complex plane. Set $z_0 = 0 + 0i$ and iterate the recursive equation $z_{n+1} = z_n^2 + c$. After a sufficient number of iterations (say about 100) if $|z_n| < 4.0$, it's probably the case that z_n will not escape and therefore c is in the Mandelbrot Set. Color point c black.

If for some *n* $|z_n| \geq 4.0$ then z_n will escape so *c* is not in the Mandelbrot Set. The color we assign to *c* depends on how fast z_n is escaping. For example, if $n \leq 10$ (z_n is escaping fast) we might color the point *c* red. If $10 < n \leq 20$ we might color c orange. If $20 < n \leq 30$ color *c* yellow, etc. So the colors surrounding the black Mandelbrot Set indicate how fast points are escaping. The colors also provide a nice aura surrounding the Mandelbrot Set.

The Geography of the Mandelbrot Set



The overall structure of the Mandelbrot Set is a central cardioid surrounded by a number of bays. The real axis bisects the Mandelbrot Set from -2.0 on the left to 0.5 on the right with the Mandelbrot Set being symmetric with respect to the real axis. Off of the bays are delicate antenna-like structures. The Mandelbrot set is connected in that given any two points in the set there is a path completely within the set connecting the two points.

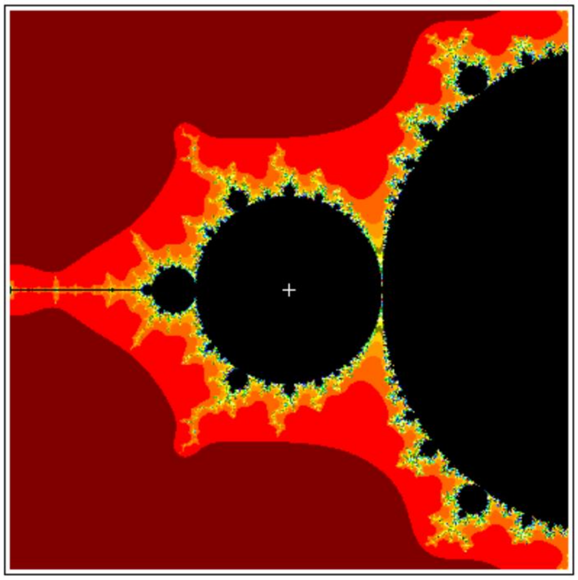
The Mandelbrot Set exhibits a high degree of symmetry. Points in the main cardioid have period 1; that is if c is any point in the central cardioid, the iteration $z_{n+1} = z_n^2 + c$ will converge to a point in the main cardioid. The large bay to the west, has period 2. If an initial c is chosen from the western bay, the iteration $z_{n+1} = z_n^2 + c$ will eventually orbit between two values. The northern and southern bays have period 3. Moving clockwise from the northern bay, the next largest north-east bay has period 4 (as does its twin to the south). The next largest has bay period 5, then period 6 and so on.

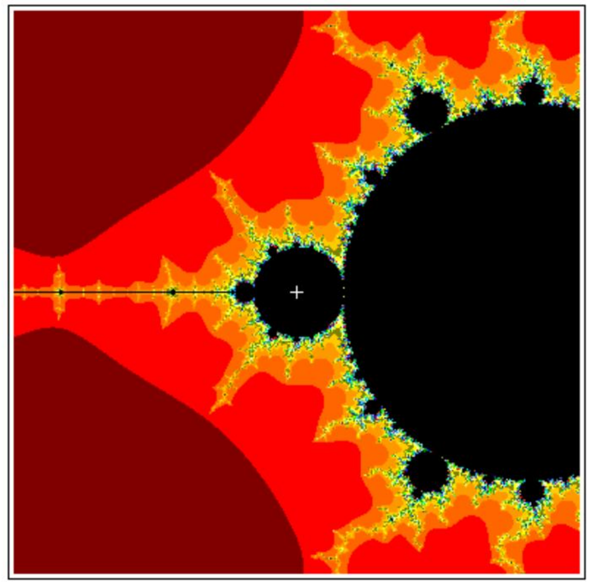
However, if you start at the northern bay and move counterclockwise, the next largest bay has period 5. And the next one period 7 etc.

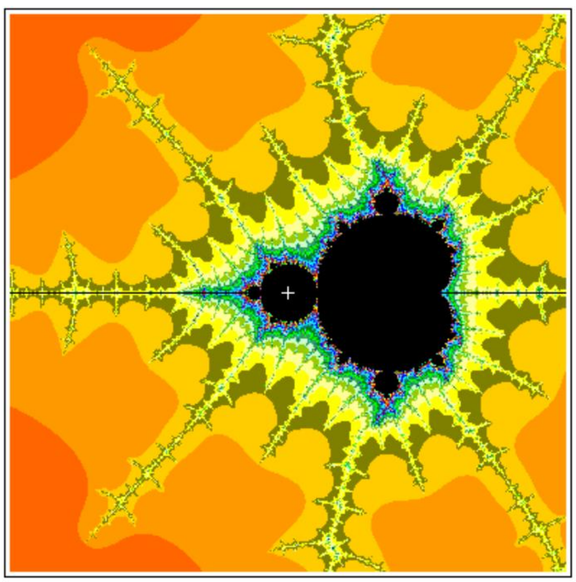
Concentrating on the western bay, its own western bay has period 4 and the next largest bay to the north (and south) has period 6. Periods are doubled.

Exploring the Western Bay

On the left a close-up of the western bay (period 2) centered at -1.0+0.0i; on the right is a close-up of the western-western bay (period 4) centered at -1.3125+0.0i. Note the small Mandelbrot Set centered along the real axis to the left off of the western-western bay and the antenna-like structures curling off the individual bays. The small back dots within the antennae like those on the real axis are smaller Mandelbrot Set like copies – satellites.







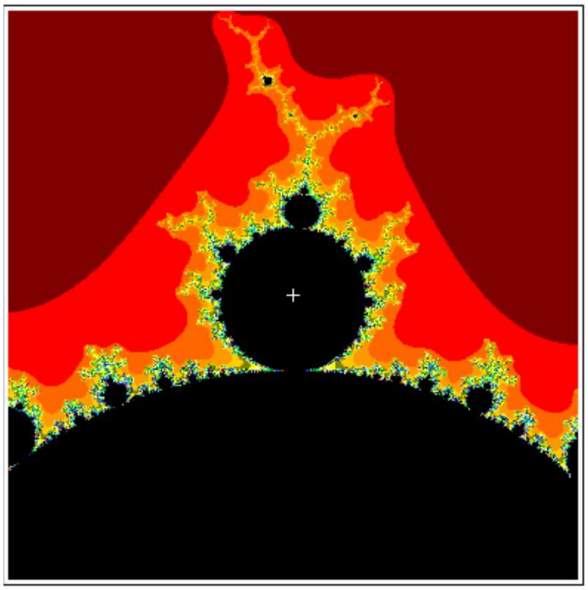
To the left is a close up of the Mandelbrot Set *island* centered on the real axis mentioned above to the left of the western-western bay at coordinates -1.48095705+0.0i.

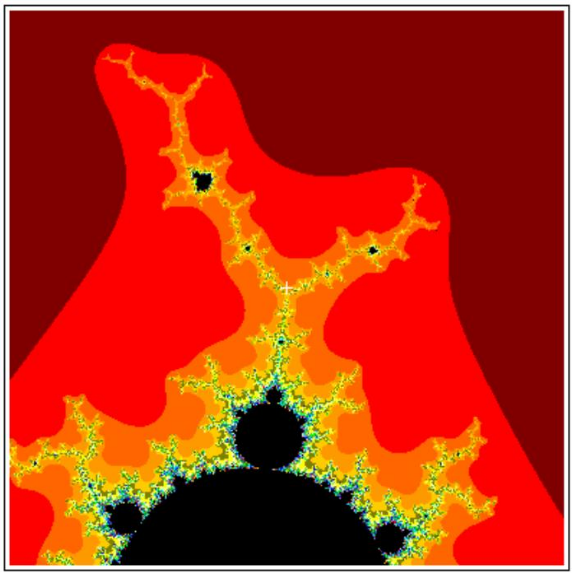
As we zoom-in we see more colors generated by the escaping values near the border of the Mandelbrot Set. Colors further up the spectrum indicate that closer a *c* value is to the Mandelbrot Set, the longer it takes to escape.

This cardioid has period 6 and its western bay has period 12. Its northern and southern bays are period 18. The western-western bay has period 24!

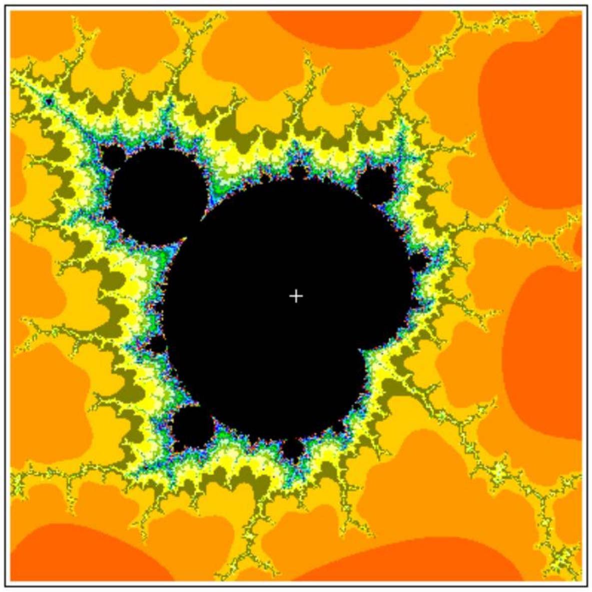
"watching mesmeric spokes form under hand:
seahorse tails, double spirals, island worlds,
crowns shrinking smaller than dust grains,
wild acid trips we mapped into eternity. "

A tour of the Northern Bay area





From the period 3 northern bay centered at -0.125+0.75i (upper left) we have a better view of the antenna-like filaments of the Mandelbrot Set. Zooming in to the where the antenna at coordinates -0.101562504+0.96093753i splits into two filaments (upper right) we see smaller satellite Mandelbrot Sets along each antenna. The one on the upper left antenna is most pronounced.

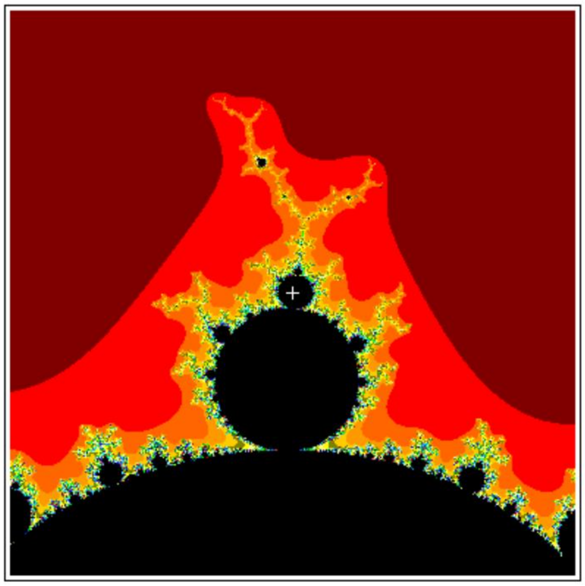


We zoom in and display the satellite Mandelbrot Set from the upper left centered at coordinates -0.15722656+1.03320312i. Again, the colors in the upper spectrum indicate the escape speed of points.

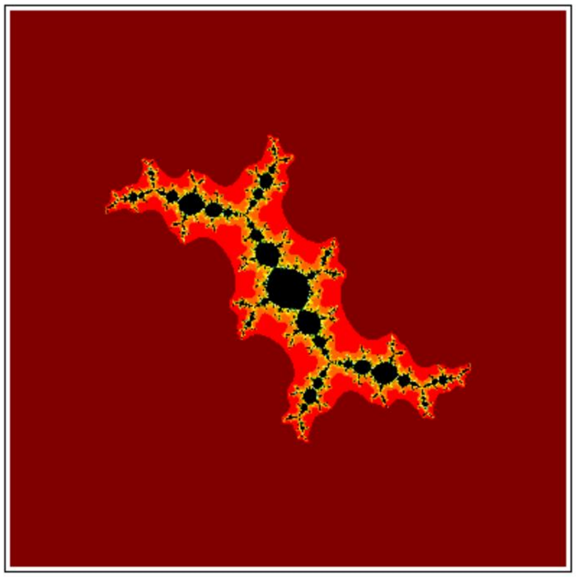
Julia Sets

The Julia Set (in the z-complex-plane) is obtained using the same $z_n = z_n^2 + c$ recurrence relation *except we fix c* and vary the initial z_0 coloring the initial point z_0 black if z_n does not escape to infinity. If z_n does escape to infinity, we color z_0 the same way we colored the Mandelbrot Set demonstrating the speed for escaping to infinity. The same algorithm is used – the only difference is that we *reverse the roles of c and* z_0.

From the Mandelbrot Set on the left we view the corresponding Julia Set. The orbit of the z values for this Julia set is 6 meaning as we iterate $z_{n+1} = z_n^2 + c$, the z values repeat after 6 iterations.



-0.11718751+0.85937504i



-0.11718751+0.85937504i

In some sense the Mandelbrot Set is like a “dictionary” for Julia Sets since every point in the Mandelbrot Set is the fixed-point *c* for the Julia Set iteration $z_n = z_n^2 + c$ where all points z that do not escape to infinity make up the Julia Set.

The Fascination of Mandelbrot and Julia Sets

The poem mentions the fascination of watching Mandelbrot and Julia sets evolve, indeed come to life on a computer screen.

Fractals were our chosen drafts of infinity,
Mandelbrot and Julia sets we’d peruse for hours,
watching mesmeric spokes form under hand:
seahorse tails, double spirals, island worlds,
crowns shrinking smaller than dust grains,
wild acid trips we mapped into eternity.

A slippery concept, Eternity
and her sister in space, Infinity,

Oversimplification

Poincaré Conjecture: Every simply connected, closed 3-dimensional manifold
is homeomorphic to the 3-dimensional sphere.

Slippery lassos try to grab forms
in a confusion of 4-D space.
Time and again, nothing to grip,
the snares slide off the backs of stubborn
balloons, billiard balls, planets, protons.
Any figure that acts like a sphere
must be a sphere.
Why acres of dusty blackboards, years,
miles of penciled chicken scrawl, a bevy of brainy folks
dreaming up patterns of flows and surgeries
to snag a proof?

Goldbach's Conjecture: Every even integer greater than two
can be expressed as the sum of two primes.

Burning brands of computer circuits
spinning for days on end
demonstrate this prime directive holds
for all integers less than 4×10^{18}
and probably soon a lot more.
But, in spite of number pyramids,
heuristic estimates, comet graphs,
reams of paper, smoky clouds of chalk,
the serious cogitations of clever folks
smoldering for near 300 years,
no proof yet.

Proposed Conjecture: Proofs of simple conjectures are not simple.

The journey begins per *una selva oscura*

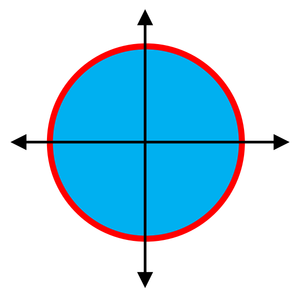
-- E R Lutken (3: A Taos Press © 2021)

Henri Poincaré (1854 – 1912) and his Conjecture

The French mathematician, Henri Poincaré, is often called the last *universalist in mathematics* due to his wide-ranging mathematical and scientific interests and results. His famous conjecture is from the area of topology which deals with asking the question of what geometric properties are preserved under continuous transformations. In other words, if you're allowed to twist, bend, stretch, push and/or pull on a geometrical object, as long as you don't cut it or put a hole in it, what properties are preserved? Topologically a triangle, a circle and a square are equivalent; they are all closed loops. Geometrically they are different.

The **Poincaré Conjecture** was solved in 2002-03 by Grigori Perelman based on the work by Richard Hamilton, another mathematician. It was one of the seven Millennial Problems in mathematics, problems selected by the Clay Mathematical Institute in 2000 as a way to direct and encourage mathematical research in the 21st century. Solutions carry a $1,000.000 reward. In 2010 Perelman was awarded the prize but he never accepted it; that's another story.

To understand the Poincaré Conjecture (it's highly technical), first consider a two dimensional disk of radius 1 centered at the origin and the circle which is its boundary or edge. The circle is a *1-sphere*, a 1-dimensional line bent back on itself. In fact, locally it looks like a line if curved somewhat. Algebraically we can describe this 1-sphere as the set



$$\{(x, y) \mid x^2 + y^2 = 1\}$$

… that is the set of all ordered pairs of numbers such that if you squared each number, they would sum to 1. If you were a tiny insect on the 1-sphere you'd think you were walking along a line. Locally it looks like a standard *Euclidean* straight line.

Now consider a three-dimensional sphere or ball of radius 1 centered at the origin and the surface of the ball. The surface is a *2-sphere*; locally it looks like a two-dimensional plane (like the surface of the earth) and algebraically we can describe this 2-sphere as

$$\{(x, y, z) \mid x^2 + y^2 + z^2 = 1\}$$

… that is, the set of all ordered triples of numbers such that if you squared each number, they would sum to 1. Again, if you were a small creature walking on the surface of the 2-sphere it would look like you're on a Euclidean plane.

So, a *3-sphere* is the surface of four-dimensional ball of radius 1 centered at the origin. The surface of a 3-sphere looks like our normal three-dimensional space (curved in a 4th dimension you cannot detect?) so if you started walking in a straight line, you would eventually return to your starting point (there is some thought that our universe, which is finite, is like a 3-sphere). Algebraically

$$\{(w,x,y,z) \mid w^2 + x^2 + y^2 + z^2 = 1\}$$

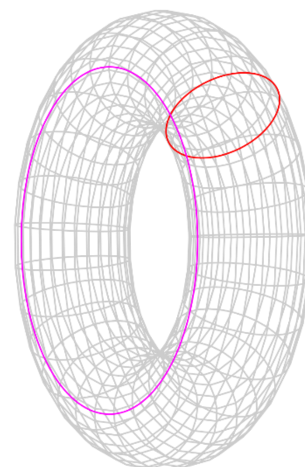
It’s hard to visualize though mathematically easy to describe.

We note that any n-sphere can have a different radius and can be centered at different a point in space; all n-spheres are *topologically equivalent*.

We can best understand the Poincaré Conjecture by looking at a *2-sphere* example.

You can take a 2-sphere (surface of a ball) and twist and pull it to deform it into other shapes just as long as you don’t cut it or “break the surface”. This is called a *homeomorphic transformation* (technically a continuous one-to-one transformation with a continuous inverse which preserves all topological properties). For example, starting with a 2-sphere you could squeeze it into a cylinder then indent the top and push down transforming it into a cup with no handle to hold water. So *topologically* the 2-dimensional sphere and a handle-less cup are the same.

However, you cannot *homomorphically* transform a 2-sphere into a donut (torus) which has a hole through it. One way to see this is any closed loop like a circle on the surface of a sphere can be squeezed down to a single point whereas a circle around the hole in the torus (in red and purple) cannot be squeezed down to a single point. That particular topological property is NOT preserved by any homeomorphic transformation.



By Krishnavedala - Own work, CC0, https://commons.wikimedia.org/w/index.php?curid=32176358

On the other hand, a torus (donut with a hole) can be deformed topologically into a cup with a handle (the handle from the donut hole) which brings to mind the old joke that a *topologist is a mathematician who cannot tell the difference between his donut and his coffee cup*.

If every closed loop can be squeezed down to a point, then the object is *simply-connected*; therefore the torus is NOT *simply connected*.

So in the *3-sphere case* the Poincaré Conjecture states that being simply-connect is all that’s required for a closed (think bounded) 3-dimensional manifold (again think a three- dimensional surface or space) to be topologically equivalent (i.e. invariant under a homeomorphic transformation) to a 3-sphere?

> “Slippery lassos try to grab forms
> in a confusion of 4-D space.
> Time and again, nothing to grip,
> the snares slide off the backs of stubborn
> balloons, billiard balls, planets, protons.
> Any figure that acts like a sphere
> must be a sphere.”

“Slippery lassos try to grab forms” is like a closed loop; “nothing to grip – the snares slide off the backs of stubborn ballons” - that can be contracted catch nothing.

Poincaré Conjecture: Every simply connected, closed 3-dimensional manifold
is homeomorphic to the 3-dimensional sphere.

Christian Goldbach (1690 - 1764) and his Conjecture

In 1752 letter written to Leonard Euler (1707 – 1783), Christian Goldbach posed the conjecture that *every even integer greater than or equal to 4 can be written as the sum of two primes*. (See *Prime Syllable Song*) For example, 12 = 5 + 7, 16 = 3 + 13 etc. It is still unproven although it's been checked by computer for integer values up to 4×10^{18}.

> "Burning brands of computer circuits
> spinning for days on end
> demonstrate this prime directive holds
> for all integers less than 4 X 10^{18} "

What is interesting is that Goldbach's Conjecture is easy to state and understand (unlike the Poincaré Conjecture) yet the conjecture defies proof.

> "smoldering for near 300 years,
> no proof yet."

In terms of Godel's Incompleteness Theorem (which states that for certain formally consistent systems of mathematics in which a certain amount of arithmetic (e.g. addition and multiplication) can be carried out, there are mathematical statements which are true but cannot be proved), it just might be that Goldbach's Conjecture is a rare example of a statement in mathematics which is true but can't be proved.

una selva oscura – "in a dark forest"

The Truel

Luc Bat for Sergio Leone

Three pairs of restless eyes
read faces, analyze grim odds,
a tic or subtle nod,
cracks within fixed facades, the fear
concealed in knitted sneers.
Thumbs tucked in bandoliers drop back,
touch hammers' tongues, enact
a lightning sequence, track pistols'
arcs, feather-triggers' pulls,
trivial options, bullets loosed,
permutations induced,
then mark the box of useless death.
Even played close to chest,
probabilistic bests just keys.
Nothing can guarantee
a gunman lives to see new skies

-- E R Lutken (3: A Taos Press © 2021)

The Good, the Bad and the Ugly
Mathematics of Modeling a Truel

The climax in the 1966 movie, "The Good, the Bad and the Ugly" is a three-way duel (or *truel*) between Blondie - "The Good" played by Clint Eastwood, Angle Eyes - "The Bad" played by Lee Van Cleef and Tuco – "The Ugly" played by Eli Wallach which is the inspiration for this essay on computing the probable outcomes for a *truel* given the independent probabilities $p_{i,j}$ the probability that gunfighter i hits (and kills?) gunfighter j. In what follows, to simplify the equations to be derived let $q_{i,j} = 1 - p_{i,j}$ be the corresponding probability that gunfighter i misses gunfighter j. Probabilities are values between 0 and 1 where for example, the probability of tossing a head with a fair coin is 0.5.

Since one outcome of a *truel* may result in only one gunfighter begin initially killed (to be continued as a duel?), we begin by modeling the simpler case for a duel. The mathematical techniques presented for the dual will be used when discussing the more complicated cases for a *truel*.

Modeling a Duel

Begin by supposing Gunfighter #1 has probability $p_{1,2}$ of hitting and killing (and probability $q_{1,2} = \left(1 - p_{1,2}\right)$ of missing) Gunfighter #2, and Gunfighter #2 has probability $p_{2,1}$ of hitting and killing (and probability $q_{2,1} = \left(1 - p_{2,1}\right)$ of missing) Gunfighter #1 . All $p_{i,j}$ probabilities are mutually independent (meaning the outcome one event should not affect the outcome of another).

There are two outcomes for Gunfighter #1 and two outcomes for Gunfighter #2 yielding four possible outcomes for a duel with their respective probabilities.

1. both are hit: $p_{1,2} \cdot p_{2,1}$
2. #1 wins: $p_{1,2} \cdot q_{2,1}$
3. #2 wins: $q_{1,2} \cdot p_{2,1}$
4. both survive $q_{1,2} \cdot q_{2,1}$

Note that the probabilities for the four outcomes sum to 1, as they should. Note that the *combined probability* of two independent events is obtained by multiplication.

Again! - To the limit

If both survive (the last of the four outcomes) there is a follow on second round again with four possible 2nd round outcomes, the outcome probabilities now being respectively …

4.1 both are hit: $(q_{1,2} \cdot q_{2,1}) \cdot p_{1,2} \cdot p_{2,1}$

4.2 #1 wins $(q_{1,2} \cdot q_{2,1}) \cdot p_{1,2} \cdot q_{2,1}$

4.3 #2 wins $(q_{1,2} \cdot q_{2,1}) \cdot q_{1,2} \cdot p_{2,1}$

4.3 both survive $(q_{1,2} \cdot q_{2,1}) \cdot (q_{1,2} \cdot q_{2,1}) = (q_{1,2} \cdot q_{2,1})^2$

These four 2nd round probabilities occur when a miss on the first round $(q_{1,2} \cdot q_{2,1})$ is followed by one of the four possible outcomes. Each outcome probability is obtained by multiplication: one outcome follows the other.

So, after *two rounds*, EITHER both are hit on the 1st round or both are hit on the 2nd round OR Gunfighter #2 is hit on the 1st round or hit on the 2nd round (and Gunfighter #1 is missed) OR Gunfighter #1 is hit on the first round or hit on the second round (and Gunfighter #2 is missed) OR both are missed on the 1st and 2nd round. That is …

1. $(p_{1,2} \cdot p_{2,1}) + (q_{1,2} \cdot q_{2,1}) \cdot (p_{1,2} \cdot p_{2,1})$ - both are hit
2. $(p_{1,2} \cdot q_{2,1}) + (q_{1,2} \cdot q_{2,1}) \cdot (p_{1,2} \cdot q_{2,1})$ - #1 wins
3. $(q_{1,2} \cdot p_{2,1}) + (q_{1,2} \cdot q_{2,1}) \cdot (q_{1,2} \cdot p_{2,1})$ - #2 wins
4. $(q_{1,2} \cdot q_{2,1})^2$ - both survive two rounds

Continuing on in this manner, if both survive the 2nd round and move on to a 3rd , 4th ,…, nth etc. round we see probability expressions (after *n* rounds) like …

$$(p_{1,2} \cdot p_{2,1}) + (q_{1,2} \cdot q_{2,1}) \cdot (p_{1,2} \cdot p_{2,1}) + (q_{1,2} \cdot q_{2,1})^2 \cdot (p_{1,2} \cdot p_{2,1}) + \ldots + (q_{1,2} \cdot q_{2,1})^n \cdot (p_{1,2} \cdot p_{2,1}) = \left(\sum_{k=0}^{n-1} (q_{1,2} \cdot q_{2,1})^k \right) \cdot (p_{1,2} \cdot p_{2,1})$$

Thus, as n increases without bound the limit of the (geometric) series $\lim_{n\to\infty}\sum_{k=0}^{n}\left(q_{1,2}\cdot q_{2,1}\right)^{k}=\frac{1}{1-q_{1,2}\cdot q_{2,1}}$

since $q_{1,2}\cdot q_{2,1}<1$.

Therefore, *in the limit* the probabilities for the first three outcomes (both are hit, Gunfighter #1 win, Gunfighter #2 wins) are given by

n.1: both are hit: $\left(p_{1,2}.p_{2,1}\right)\cdot\sum_{k=0}^{n}\left(q_{1,2}\cdot q_{2,1}\right)^{k}\Rightarrow\frac{\left(p_{1,2}\cdot p_{2,1}\right)}{1-q_{1,2}\cdot q_{2,1}}$

n.2: #1 wins: $\left(p_{1,2}.q_{2,1}\right)\cdot\sum_{k=0}^{n}\left(q_{1,2}\cdot q_{2,1}\right)^{k}\Rightarrow\frac{p_{1,2}\cdot q_{2,1}}{1-q_{1,2}\cdot q_{2,1}}$

n.3: #2 wins: $\left(q_{1,2}.p_{2,1}\right)\cdot\sum_{k=0}^{n}\left(q_{1,2}\cdot q_{2,1}\right)^{k}\Rightarrow\frac{q_{1,2}\cdot p_{2,1}}{1-q_{1,2}\cdot q_{2,1}}$

while the probability of both surviving is given by

n.4: both survive $\left(q_{1,2}\cdot q_{2,1}\right)^{k}\Rightarrow 0$ since $q_{1,2}\cdot q_{2,1}$ being less than 1, the limit goes to 0.

Some Examples

To get a feel for the above *Dual* calculations suppose both gunfighters have a 50-50 chance of hitting the other; that is $p_{1,2}=0.5$ (so $q_{1,2}=1.0-p_{1,2}=0.5$) and $p_{2,1}=0,5$ (so $q_{2,1}=1.0-p_{2,1}=0.5$). Then plugging these numbers into the four outcome equations ….

n.1 both are hit: $\frac{\left(p_{1,2}\cdot p_{2,1}\right)}{1-q_{1,2}\cdot q_{2,1}}=\frac{0.5\times 0.5}{1-0.5\times 0.5}=\frac{0.25}{0.75}=\frac{1}{3}$

n.2 #1 wins: $\frac{p_{1,2}\cdot q_{2,1}}{1-q_{1,2}\cdot q_{2,1}}=\frac{0.5\times 0.5}{1-0.5\times 0.5}=\frac{0.25}{0.75}=\frac{1}{3}$

n.3 #2 wins: $\frac{q_{1,2}\cdot p_{2,1}}{1-q_{1,2}\cdot q_{2,1}}=\frac{0.5\times 0.5}{1-0.5\times 0.5}=\frac{0.25}{0.75}=\frac{1}{3}$

n.4 both survive: 0 probability

In retrospect this result makes sense.

However, suppose Gunfighter #1 is slightly better than Gunfighter #2 – say Gunfighter #1 has probability $p_{1,2}=0.6$ ($q_{1,2}=1.0-p_{1,2}=0.4$) while Gunfighter #2 has the same probability given above $p_{2,1}=0,5$ (so $q_{2,1}=1.0-p_{2,1}=0.5$). Then plugging these numbers into the four outcome equations ….

n.1 both are hit: $\frac{\left(p_{1,2}\cdot p_{2,1}\right)}{1-q_{1,2}\cdot q_{2,1}}=\frac{0.6\times 0.5}{1-0.4\times 0.5}=\frac{0.3}{0.8}=\frac{3}{8}>\frac{1}{3}$

n.2 #1 wins: $\frac{p_{1,2} \cdot q_{2,1}}{1-q_{1,2} \cdot q_{2,1}} = \frac{0.6 \times 0.5}{1-0.4 \times 0.5} = \frac{0.3}{0.8} = \frac{3}{8} > \frac{1}{3}$

n.3 #2 wins: $\frac{q_{1,2} \cdot p_{2,1}}{1-q_{1,2} \cdot q_{2,1}} = \frac{0.4 \times 0.5}{1-0.4 \times 0.5} = \frac{0.2}{0.8} = \frac{1}{4} < \frac{1}{3}$

n.4 both survive: 0 probability

So Gunfighter #1 has a slightly better chance of winning ($\frac{3}{8} vs \frac{1}{3}$), Gunfighter #2 has a slightly smaller chance of winning ($\frac{1}{4} vs \frac{1}{3}$) while the chance of both losing increases ($\frac{3}{8} vs \frac{1}{3}$).

Modeling a Truel

A *truel* which is more complicated can start in one of two possible configurations assuming of course all three gunfighters are unable to shoot at two targets at the same time.

Without loss of generality, we can say either Gunfighter #1 shoots at Gunfighter #2 who shoots at Gunfighter #3 who shoots at Gunfighter #1 (a sort of *round robin* face off), or Gunfighter #1 and Gunfighter #2 face off in a dual and Gunfighter #3 shoots at either #1 or #2 (a *dual plus one*).

However, unlike a duel where the only option is for the gunfighters to shoot at the other, in a *truel* the first round could either be a round robin or a duel plus one where any gunfighter shooting at any other gunfighter. Furthermore, if there are no casualties after the first round, the gunfighters could switch targets which might result in a different configuration of round robin or dual plus one. There are more variables (and probabilities) to deal with than with a dual.

Coding a Monte-Carlo based Truel?

The mathematical approach used above to model a *dual* will not work to model a *truel* in that one needs to factor in the probabilities of round robin and/or duel plus one face-offs (randomly?) occurring in an actual gunfight especially if early rounds resulted in no casualties. Better is a Monte Carlo approach which would randomly allow each gunfighter to randomly select a target. First let's consider how a *Monte Carlo duel* works.

Monte Carlo Duel

Programming a Monte Carlo duel uses the two probabilities $p_{1,2}$ and $p_{2,1}$ along with a random number generator to *simulates a duel.* Outcomes are *random* but subject to the constraints imposed by the two probabilities $p_{1,2}$ and $p_{2,1}$. If a simulation is executed many times and the results recorded (the more the better) it should approach what would actually happen and more or less agree with the non-Monte Carlo method covered above. Hence the number of rounds N to execute is also input to the simulation.

For each round the random number generator is used to generate a random number between 0 and 1 for each gunfighter and if that numbers is l*ess than or equal to the probability* $p_{i,j}$ a corresponding hit is recorded. A round continues until one or both gunfighters are eliminated and the running counts for no survivors, #1 wins, or #2 wins are incremented. (It is assumed that a round of gunfire continues until someone is hit which stops that round.) When the N rounds are completed, each running count is divided by N to yield the corresponding probabilities of no survivors, #1 hits, or #2 hits. Given a large enough initial value for N, the resulting probabilities will be (should be) very close to the analytic mathematical results calculated above. No surprise there!

To demonstrate this, here is the Python code to execute a Monte Carlo Duel. Comments explaining the code are in red.

```
# Desc: Monte Carlo Simulation of a Duel

from random import *  # access random number generator functions
                      # random() generates a random value between 0.0 and 1.0
def hit(p):
#
#   p is probablity of a hit
#
#   Returns True if random() <= p  - that is a hit!
#   Otherwise returns False - a miss

    return (random() < p)

def main():   # main program code

    print("\nMonte Carlo Duel\n")

    # input probabilities of a hit and number of rounds to execute

    p1 = eval(input("Input Gunfighter #1 probability of a hit: "))
    p2 = eval(input("Input Gunfighter #2 probability of a hit: "))
    N = eval(input("Input Total Number of Rounds (Trials): "))

    cnt0 = 0 # Count of both Gunfighters hit
    cnt1 = 0 # Count of Gunfighter #1 wins
    cnt2 = 0 # Count of Gunfighter #2 wins

    RoundNumber = 0  # running count of number of completed rounds

    while (RoundNumber < N):
        shot1 = hit(p1)  # Gunfighter #1 shoots with probability p1
        shot2 = hit(p2)  # Gunfighter #2 shoots with probability p2
        if (shot1 and not shot2):  # Gunfighter #1 hits, #2 misses
            cnt1 = cnt1 + 1        # increment Gunfighter #1 count
            RoundNumber = RoundNumber + 1 # increment count of rounds
        elif (not shot1 and shot2): # Gunfighter #1 misses, #2 hits
            cnt2 = cnt2 + 1          # increment Gunfighter #2 count
            RoundNumber = RoundNumber +1 # increment count of rounds
        elif (shot1 and shot2): # both Gunfighters miss
            cnt0 = cnt0 + 1          # increment count for both are hit
            RoundNumber = RoundNumber + 1 # increment count of rounds

    # Display Results nicely formatted

    print("\nTotal Gunfighter #1 Wins = {0} : {1:5.2f}%".format(cnt1,cnt1/N*100))
    print("Total Gunfighter #2 Wins = {0} : {1:5.2f}%".format(cnt2,cnt2/N*100))
    print("No survivors = {0} : {1:5.2f}%".format(cnt0,cnt0/N*100))

main()
```

Below we executed the program twice, the first time using the probabilities $p_{1,2} = 0.5$ for Gunfighter #1 and $p_{2,1} = 0.5$ for Gunfighter #2. After 50,000 rounds the figures closely match the result obtained from the previous mathematical calculations (as they should). For the second run we used probabilities $p_{1,2} = 0.6$ for Gunfighter #1, $p_{2,1} = 0.5$ for Gunfighter #2 and again for 50000 rounds. Again, the results closely match those obtained from mathematical calculations. Due to the built in use of a random number generator, there will always be some variation (which decreases as N, the number of rounds played increases).

```
Monte Carlo Duel

Input Gunfighter #1 probability of a hit: 0.5
Input Gunfighter #2 probability of a hit: 0.5
Input Total Number of Rounds (Trials): 50000

Total Gunfighter #1 Wins = 16569 : 33.14%
Total Gunfighter #2 Wins = 16743 : 33.49%
No survivors = 16688 : 33.38%

Monte Carlo Duel

Input Gunfighter #1 probability of a hit: 0.6
Input Gunfighter #2 probability of a hit: 0.5
Input Total Number of Rounds (Trials): 50000

Total Gunfighter #1 Wins = 18805 : 37.61%
Total Gunfighter #2 Wins = 12454 : 24.91%
No survivors = 18741 : 37.48%
```

Monte Carlo Truel

A Monto Carlo Truel is more complicated just as a *truel* is more complected. Here you have six probabilities $p_{i,j}$ for $i, j = 1, 2, 3$ plus three other probabilities $s_{2,3}$, $s_{1,2}$, and $s_{3,1}$ being the probabilities that gunfighter #1 chooses to shoot at gunfighter #2, gunfighter #2 chooses to shoot at gunfighter #3 and gunfighter #3 chooses to shoots at gunfighter #1 (with $s_{2,1} = 1 - s_{1,2}$, $s_{3,2} = 1 - s_{2,3}$, and $s_{1,3} = 1 - s_{3,1}$ being the complementary probabilities). Initial input values for a program are the *nine* probabilities plus N the number of rounds. When one gunfighter is eliminated the code for Monte Carlo duel can be used to decide the outcome which negates the need for the corresponding $s_{i,j}$ probability. There are four possible outcomes with the final probable outcomes computed in the obvious way.

See the *Programming Appendix* for a Python program that implements Monte Carlo Truel.

A Brief History on the Origins of Monte Carlo Techniques

The Monte Carlo method uses repeated random sampling to obtain numeric results (using computer generated pseudo-random numbers) when deterministic calculations are too complicated to provide an answer. The initial work was done by John von Neuman, Nicholas Metropolis, and Stanislaw Ulam in the late 40's at the Los Alamos Lab to simulate and understand the process of neutron diffusion (in the process of a nuclear chain reaction).

According to von Neuman and Ulam the idea originated from Ulam's interest in random processes. To quote Ulam "The procedure is analogous to playing a series of solitaire cards games and is performed on a computing machine. It requires, among others, the use of random numbers with a given distribution."

The name "Monte Carlo" was suggested by Metropolis referring to the Monte Carlo casino in Monaco where Ulam's uncle would borrow money from relatives to gamble.

Monte Carlo methods require the use of a random number generator to generate the many random values used which lead von Neumann to observe …

> *"Anyone who considers arithmetical methods of producing random digits is, of courses, in a state of sin"* – J. von Neumann

… since computers are deterministic machines.

More Monte Carlo - Monte Carlo Pi?

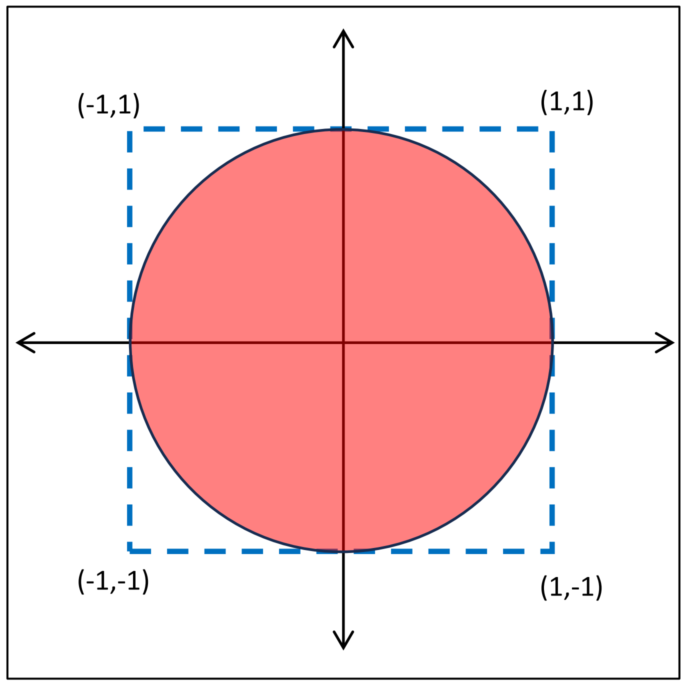
Consider a circle of radius 1 centered at the origin of the xy-plane enclosed in a 2x2 square box (see diagram on right). Since the radius of the circle is 1, the area of the circle is

$$Area = \pi r^2 = \pi$$

and the area of the surrounding 2x2 square is 4.

For a large integer *n*, *randomly generate n* (x,y) coordinate pairs such that $-1 \le x \le +1$ and $-1 \le y \le +1$ counting number of (x,y) pairs (or points) that are in the circle; that is such that $\sqrt{x^2 + y^2} \le 1$. If *k* is the number of xy pairs in the circle, then $\frac{k}{n} \times 4$ is a *rough estimate* for the area of the circle; that is $Area \approx \frac{k}{n} \times 4$

Therefore $\pi \approx \frac{k}{n} \times 4$.



The Python program below implements a Monte Carlo simulation to generate pi.

```
from random import *

def main():

    print("\nMonte Carlo Pi\n")
    n = eval(input("Enter number of rounds: "))
    k = 0  # count of points in circle

    for i in range(n):
        x = 2.0 * random() - 1.0
        y = 2.0 * random() - 1.0
        r = x*x + y*y  # is the point (x,y) in the circle?
        if r <= 1.0:
            k = k + 1

    pi = (k/n)*4.0
    print("\npi ~ {0:6.4f}\n".format(pi))

main()
```

A Sample Run

```
Monte Carlo Pi

Enter number of rounds: 100000

pi ~ 3.1348
```

Back to the Movie

In the 1966 movie both Tuco and Blondie aim at Angel Eyes who is killed by Blondie (a duel plus one); Tuco's gun jams as he tries to shoot Angel Eyes. After being hit Angel Eyes tried to shoot Blondie but fails. Blondie does not engage Tuco.

Math History in a Few Bad Clerihews

Brahmagupta
Worth the hoopla
In Dark Ages of Gothic invasions
He was solving quadratic equations

Blaise Pascal
True rascal
Worked hard to entangle
That fucking triangle

Sir Isaac Newton
Partial towards gluten
Known for plummeting apples, flummoxing wigs
And orbital motions of cookies with figs

George Boole
No fool
But batshit demagogic
On the subject of logic

Ada Lovelace
Ignored her place
And programmed a computer
That's smarter than you are

Bernhard Riemann
Straight up demon
Guilty of the heinous crimes
Of a cluster-fuck of primes

L. E. J. Brouwer
Often quite dour
But cheered us up, since, after all
He figured out the hairy ball

Edward Lorenz
Bought a Mercedes Benz
Sadly, the initial conditions
Caused expensive chaotic collisions

-- E R Lutken (3: A Taos Press © 2021)

Brahmagupta (598-670)

Today the quadratic *equation* $0 = ax^2 + bx + c$ can be solved using the quadratic *formula*

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

The ancient Babylonian were able to solve some forms of the quadratic equation but lacked the algebraic notation needed for the general solution. (Algebraic notation wouldn't appear until the 16th or 17th centuries CE). Apparently, the Greeks whose mathematical approach and methods were geometric had some similar results but again were hampered by the lack of algebraic notation.

However, the Indian mathematician Brahmagupta who also lacked algebraic notation was able to derive a generalized method for solving a quadratic equation of the form $ax^2 + bx = c$ (not $0 = ax^2 + bx + c$). That is

"To the absolute number" (c?)"multiplied by four times the [coefficient of the] square, add the square of the [coefficient of the] middle term; the square root of the same, less the [coefficient of the] middle term, being divided by twice the [coefficient of the] square is the value." (*Brahmasphutasiddhanta*, Colebrook translation, 1817, page 346) This is equivalent to

$$x = \frac{\pm\sqrt{4ac + b^2} - b}{2a}$$

Brahmagupta is also the first known mathematician to have treated zero as a number and not just as a place holder. See *Zero*.

Blaise Pascal (1623-1662)

Although not the first to make use of what has become known as Pascal's Triangle, he made use of it solving problems involving probability.

A property of Pascal's Triangle is that any row can be obtained by summing adjacent entries from the previous row – for example 4 + 6 = 10.

If we number the rows *n* starting at 1 and the diagonals (left to right) k starting at 0, then the (n, k) entry is given by the binomial coefficient $\binom{n}{k} = \frac{n!}{k!(n-k)!}$ recalling that by definition 0!=1 which is the number of possible combinations (sets) obtain by choosing k items from a set of n. This is where Pascal's Triangle is used in probability. For example, $\binom{n}{k} p^k \cdot q^{n-k}$ is the probability of *k* successes (probability of success = *p*) out of *n* independent trials (probability of failure = *q* = *1* – *p*). For

```
            1  1
          1   2   1
        1   3   3   1
      1   4   6   4   1
    1   5   10  10  5   1
  1   6   15  20  15   6  1
1   7   21  35  35  21   7   1
1   8   28  56  70  56  28   8   1
```

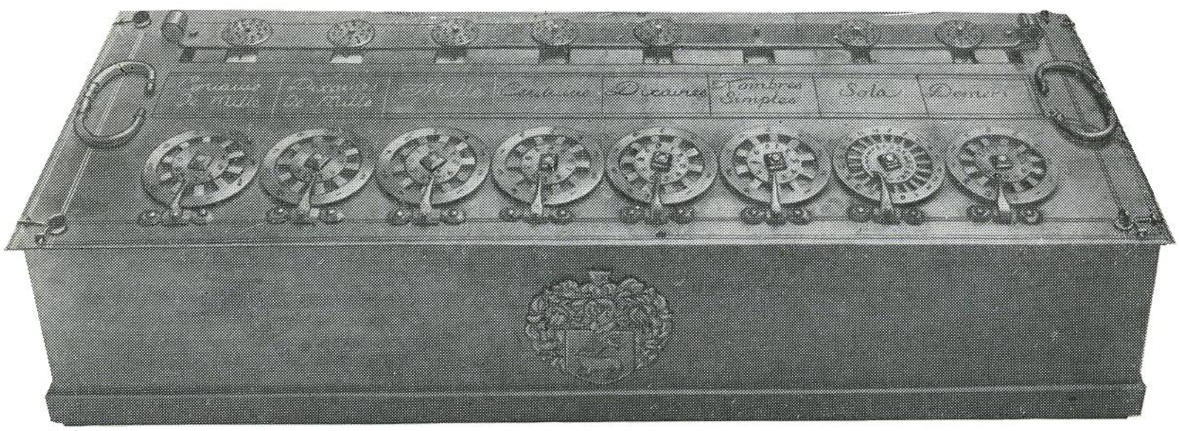
example, if you roll a fair die 5 times (or roll 5 dice one time) the probability you will roll *exactly* three 6's is

$$\binom{5}{3}\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^2 = 10\cdot\frac{1}{216}\cdot\frac{25}{36} = 0.03215$$

Pascal's Triangle and the Binomial Theorem: The entries in each row of Pascal's Triangle are the coefficients for the expansion of the binomial $(a+b)^n$. For example, the 3rd row integers 1 3 3 1 are the coefficients for the expansion of $(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3$. In general, $(a+b)^n = \sum_{k=0}^{n}\binom{n}{k}a^k b^{n-k}$

Pascal also invented a mechanical device, the *Pascaline* (see below), that could add and subtract making him one of the first calculating machine inventors. Unfortunately, the tooling technology of the 17th century wasn't quite up to the manufacturing standards needed by the *Pascaline*, so the machine was not a success.

Subsequently the computer language Pascal was named after him.



Pascaline made for French currency which once belonged to Louis Perrier, Pascal's nephew. The least significant denominations, sols and deniers, are on the right. By J. A. V. Turck - Downloaded 2008-1-9 from J. A. V. Turck (1921) Origin of Modern Calculating Machines, Western Society of Engineers, Chicago, USA, p.10, fig.1 from Google Books, Public Domain, https://commons.wikimedia.org/w/index.php?curid=3393262

Isaac Newton (1642-127)

Along with Leibniz, Newton is considered one of the inventors (or discoverers ?) of the calculus which he used to describe what is called Newtonian mechanics. His law of gravity explained planetary motion. See *Fundamental of Mathematics* and π.

Aside from his work with the calculus and planetary motion, Newton also discovered the *Generalized Binomial Theorem* which extended the Binomial Theorem (see above) to non-integer values s. That is

$$(1+x)^s = \sum_{k=0}^{\infty}\binom{s}{k}x^k$$

where $\binom{s}{0}=1$, $\binom{s}{1}=s$, $\binom{s}{2}=\frac{s(s-1)}{2!}$ and in general $\binom{s}{k}=\frac{s(s-1)(s-2)...(s-k+1)}{k!}$.

It's not difficult to see that if *s* is a positive integer *n*, the standard Binomial Theorem pops out since all terms after the n^{th} are zero.

Aside: The *Fig Newton* was named after a town in Massachusetts.

George Boole (1815-1864)

His laws of thought led to the branch of mathematics (and logic) called Boolean Algebra which among other things is used to design computers.

Boolean algebra is two-valued: True or False – often represented by 1 for True and 0 for False. The standard operations are AND, OR and NOT which work the way they do in everyday English. For example, if x and y are statements (like x = "Today is Monday" and y = "It's snowing") then x AND y is True if and only if both statements x and y are True. The compound statement x OR y is True if and only if either x or y or both x and y are True. And obviously NOT x negates the truth of x. All in all, it's simple yet Boolean algebra can be used to design computer circuits – and computers.

Using `1` for True and `0` for False, Truth Tables are used to define the values derived from the Boolean operations AND, OR and NOT

a	b	a AND b
0	0	0
0	1	0
1	0	0
1	1	1

a	b	a OR b
0	0	0
0	1	1
1	0	1
1	1	1

a	NOT a
0	1
1	0

The Boolean operations of AND, OR and NOT have corresponding digital gate symbols (right) that can be used to design circuits that implement Boolean algebra expressions.

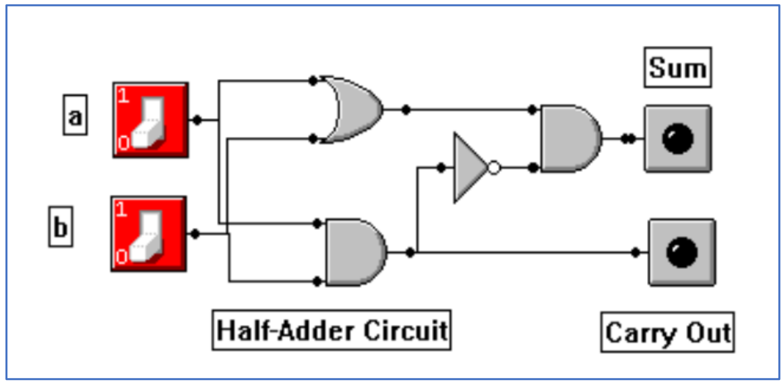
Truth Tables can be used to evaluate compound Boolean expressions. For example, using a Truth Table consider the Boolean expression (a OR b) AND (NOT (a AND b)) for all possible input values of a and b

a	b	a OR b	a AND b	NOT (a AND b)	(a OR b) AND (NOT (a AND b))
0	0	0	0	1	0
0	1	1	0	1	1
1	0	1	0	1	1
1	1	1	1	0	0

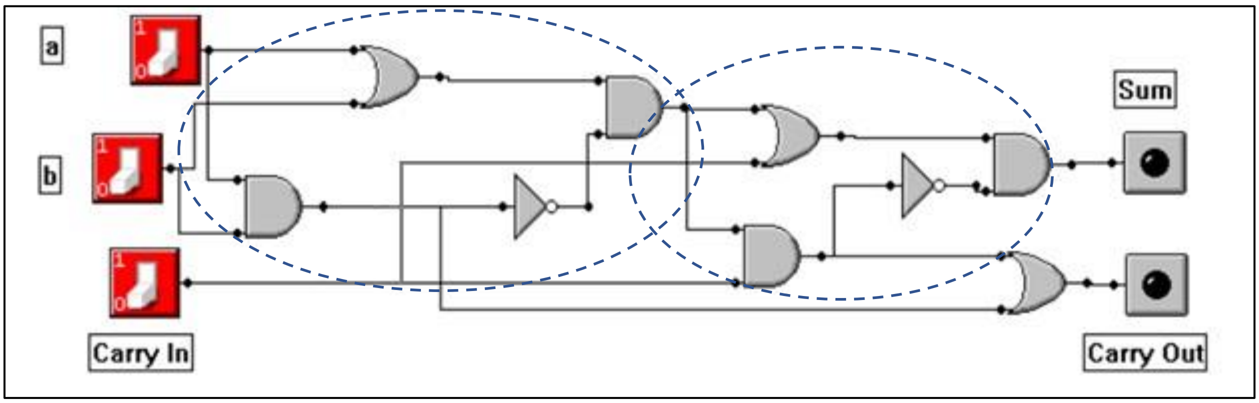
Using the results from the above compound Boolean expression (a OR b) AND (NOT (a AND b)) as a *sum bit* and using (a AND b) as a *carry bit* we have four rules for the addition of two binary integers: thus, Boolean algebra to arithmetic – the resulting circuit called a Half-Adder!

a	b	(a OR b) AND (NOT (a AND b))	a AND b	
0	0	0	0	0 + 0 = 0
0	1	1	0	0 + 1 = 1
1	0	1	0	1 + 0 = 1
1	1	0	1	1 + 1 = 10

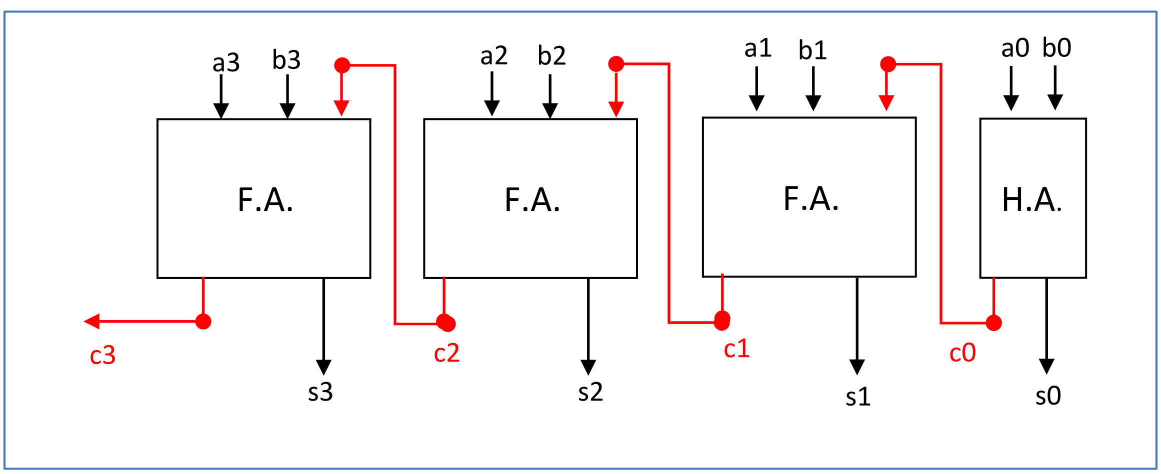
```
 0     0     1     1
+0    +1    +0    +1
---   ---   ---   ---
00    01    01    10
```



To the left is the corresponding digital circuit for a Half-Adder. Below if we *cascade* two Half Adders (contained in dashed circles) together we have a Full Adder with three inputs: a, b, and carry in and two outputs, sum and carry out. The final carry-out is the OR of the two carry outs from the Half Adders.



Finally, if a Half Adder is *cascaded* with 3 Full Adders (see below) a 4-bit Ripple Carry Adder can be constructed which will add any two 4-bit binary integers. Of course, why stop at 4?



Ada Lovelace (1815-1852)

Ada Byron Lovelace was the daughter of the poet Lord George Gordon Byron. Her parents separated after a year of marriage with Byron leaving England for good one month after Ada was born, never to see his daughter again. Ada's mother Anne (né Anne Isabella Milbanke) was determined that Ada would have nothing to do with her father or with *poetry*, so it was that Ada was introduced early in life to the study of *mathematics*.

Meeting Charles Babbage she saw a demonstration of a model of his Difference Engine which performed calculations using the method of divided differences (see *Elegy for a Slide Rule*). She also became acquainted with Babbage's Analytic Engine to the extent that she was able to write programs for it; thus, she is considered to be the world's first programmer and has the Ada programming language named after her. Early on she saw the possibilities of computation as realized in the Analytic Engine.

"The distinctive characteristic of the Analytical Engine, and that which has rendered it possible to endow mechanism with such extensive faculties as bid fair to make this engine the executive right-hand of abstract algebra, is the introduction into it of the principle which Jacquard devised for regulating, by means of punched cards, the most complicated patterns in the fabrication of brocaded stuffs. It is in this that the distinction between the two engines lies. Nothing of the sort exists in the Difference Engine. We may say most aptly that the Analytical Engine weaves algebraical patterns just as the Jacquard loom weaves flowers and leaves."

Bernhard Riemann (1826-1866)

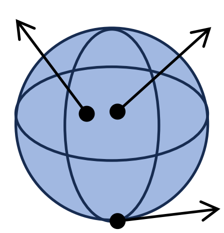
Riemann is known for his work in Geometry, the Riemann Integral (taught in introductory calculus courses), and the Zeta Function which is tied to determining the number of primes less or equal to a given integer. The Riemann Hypothesis (unproved) states that all zeroes of the complex valued zeta function

$$\zeta(s) = \sum_{n=0}^{\infty} \frac{1}{n^s} = \prod_{p\ prime} \frac{1}{1-p^{-s}}$$

lie on the line $z = \frac{1}{2}$. The Riemann Hypothesis is currently the greatest open question in mathematics and closely connected to computing the value of $\pi(n)$, the prime counting function (recall that $\pi(n)$ is the number of primes less than or equal to *n*). Notice in the above zeta function equation that we're taking a product (the Π symbol) over all the primes.

L.E.J. Brouwer (1881-1966)

Brouwer is known for the famous Fixed Point Theorem which states that for any continuous function $f(x)$ mapping a convex compact set to itself (for example, a closed line interval, a closed disk, or a sphere), there is a point x_0 such that $f(x_0) = x_0$. See *Distillations*.



The *hairy ball theorem* which is similar to Brouwer's theorem states that if on the surface of a sphere you have a continuous vector field (think of arrows attached to each point on the sphere) somewhere an arrow *points straight up*. In other words, "you can't comb a hairy ball flat without creating a cowlick."

In other words, mathematical objects (like hairy spheres) must behave in certain ways.

Edward Lorenz (1917-2008)

Edward Lorenz, mathematician and metrologist, is known for his pioneering work in numerical weather modeling and chaos and complexity theory. Chaotic systems are sensitive to initial conditions (the so-called *butterfly effect*) meaning small differences in the beginning lead to large differences in long-term behavior. It's why long-term weather prediction is not possible.

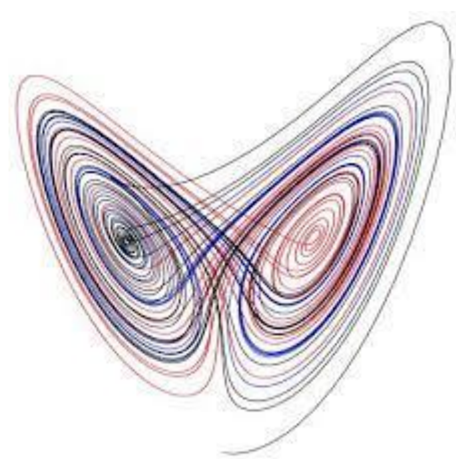
In the 1950's Lorenz begin using numerical methods to model weather and weather predictions – using variables for wind, pressure, temperature etc. Then in 1961 Lorenz using a simple computer (Royal McBee LGP-30) to model weather simulation made a startling discovery which he explained in his book *The Essence of Chaos*

> *"At one point I decided to repeat some of the computations in order to examine what was happening in greater detail. I stopped the computer, typed in a line of numbers that it had printed out a while earlier, and set it running again. I went down the hall for a cup of coffee and returned after about an hour, during which time the computer had simulated about two months of weather. The numbers being printed were nothing like the old ones. I immediately suspected a weak vacuum tube or some other computer trouble, which was not uncommon, but before calling for service I decided to see just where the mistake had occurred, knowing that this could speed up the servicing process. Instead of a sudden break, I found that the new values at first repeated the old ones, but soon afterward differed by one and then several units in the last decimal place. . . . The numbers I had typed in were not the exact original numbers, but were the rounded off values that had appeared in the original printout. The initial round-off errors were the culprits; they were steadily amplifying until they dominated the solution. In today's terminology, there was chaos."*

As mentioned above, sensitivity to initial conditions means that even a very small difference in the initial values for a model will lead in the long run to very large differences; thus, long term accurate prediction of weather is impossible as is also the case for other systems which are *chaotic*.

Lorenz also discovered a set of mathematical objects called *strange attractors*, systems which are chaotic yet have a structure to them. For example, the set of equations below where the space coordinates x, y, and z are all functions of the variable t for time and σ, ρ, and β are parameters will generate the three-dimension Lorenz Attractor (right) for certain initial values for the parameters σ, ρ, and β.

$$\frac{dx}{dt} = \sigma(y - x)$$
$$\frac{dy}{dt} = x(\rho - z) - y$$
$$\frac{dz}{dt} = xy - \beta z$$



The Lorenz Curve (an object in 3-space – the picture on the right is a projection onto a plane) is called a *strange attractor* because no matter what the initial values for x, y and z are, eventually the orbit of the points generated will end up spiraling (?) around and around the Lorenz Attractor. However, even if two starting points (values) for x, y, and z are initially very close, because of *sensitivity of initial conditions*, their orbits will appear at different points on the attractor; not close together.

Sensitivity to initial conditions is often referred to as the *Butterfly Effect*. Although the Lorenz Attractor *does look a bit like a butterfly*, the *Butterfly Effect* refers to weather's sensitivity to initial conditions where the beating of a Butterfly's wing in the Amazon might affect the weather conditions in the United States.

Or maybe because when …

Edward Lorenz
Bought a Mercedes Benz
Sadly, the initial conditions
Caused expensive chaotic collisions.

For more on Chaos see *All is Number*

Distillations

Vivid sand wheel
Rim of charnel ground
Sacred core

Every continued function
from a closed/bounded disk
to itself has at least one fixed point
where F(y) = y

Stone pillars
Vaulted canopy
Shafts of light
Whispers living in air
The worn step, the candle-lit altar

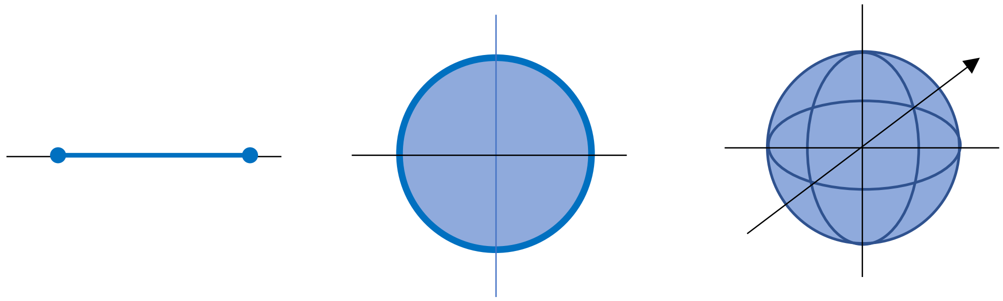
Multifaceted Eulerian form
partitioned into infinite
vertices, edges, faces
yielding two still points
V - E + F = 2

-- E R Lutken (3: A Taos Press © 2021)

The Brouwer Fixed Point Theorem!

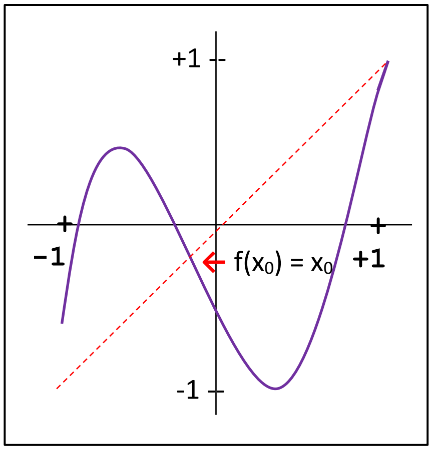
The Brouwer Fixed Point Theorem (see *Math History in a Few Bad Clerihews*) states that any continuous function $y = f(x)$ from a closed bounded disk onto itself has a fixed point; that is, there is a point x_0 on the closed disk such $f(x_0) = x_0$. Let's explore this in some detail.

In one dimension, the line (?) $\{x \mid -1 \le x \le 1\}$ would be a closed bounded *linear* disk. In two dimensions, the unit circle (disk) $\{(x, y) \mid x^2 + y^2 \le 1\}$ is a closed bounded *flat* disk - closed since we're including its boundary perimeter. In three dimensions the solid unit ball $\{(x, y, z) \mid x^2 + y^2 + z^2 \le 1\}$ would be a closed bounded *solid* disk.



Note that the radius can be any no-zero value; here we've chosen 1 for convenience.

The Brouwer Fixed Point Theorem can be hard to see in the case of the 2-D flat disk or the 3-D solid ball but it's easier to see with the 1-D line $\{x \mid -1 \le x \le 1\}$ where $y = f(x)$ maps the closed interval [-1,1] on the x-axis continuously onto the closed interval [-1,1] on the y-axis



The purple curve on the right is a graph for a continuous function from the interval [-1,1] onto [-1,1] where every point on the x-axis from -1 to 1 is mapped onto a point between -1 and 1 on the y-axis. Indeed, sometimes a y-value can have two or more x-values mapped to it. Since the curve is continuous there are no holes of breaks in the curve. The red dotted line is the line y = x. Note that it must cut the purple curve and where it does, this is where $f(x_0) = x_0$ - a demonstration of the Brouwer Fixed Point Theorem for the one-dimensional case.

Algebraically we can prove the same.

Let $y = f(x)$ be a continuous onto function from $[-1,1]$ to $[-1,1]$ (i.e. $f:[-1,1] \to [-1,1]$). Consider the function $g(x) = x - f(x)$. Note that being the difference of two continuous functions, $g(x)$ is *continuous*.

Case 1: If $f(-1) = -1$ then $g(x)\big|_{x=-1} = x - f(x)\big|_{x=-1} = 0$ since $f(-1) = -1$ and therefore -1 is a fixed point for $f(x)$.

Case 2: If $f(1) = 1$ then $g(x)\big|_{x=1} = x - f(x)\big|_{x=1} = 0$ since $f(1) = 1$ and therefore 1 is a fixed point for $f(x)$

Case 3: $f(-1) > -1$ and $f(1) < 1$. Therefore $g(-1) = -1 - f(-1) < 0$ and $g(1) = 1 - f(1) > 0$. Since $g(x) = x - f(x)$ is a continuous function on the closed interval and $g(-1) = 1 - f(-1) < 0$ and $g(1) = 1 - f(1) > 0$ on the closed interval $[-1,1]$ then by the *Intermediate Value Theorem* for continuous functions on a closed interval, there is a point x_0 on the interval $(-1,1)$ such that $g(x_0) = x_0 - f(x_0) = 0$ or $x_0 = f(x_0)$ making x_0 a fixed point.

"Every continued function
from a closed/bounded disk
to itself has at least one fixed point
where F(y) = y"

Aside: *The Intermediate Value Theorem* says that if the graph of a continuous function $y = f(x)$ on a closed interval is negative at one endpoint and positive at the other endpoint, then the graph must cross the x-axis; that is, somewhere between the two endpoints there is a point x_0 such that $f(x_0) = 0$. It is intuitively obvious, but it does require a formal proof. In short, continuous functions are well-behaved!

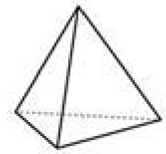
Euler Polyhedron Formula: V – E + F = 2

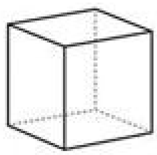
Euler's Polyhedron Formula (see *All is Number*) expresses an invariant relationship between the number of vertices V, the number of edges E, and the number of faces F for a 3-dimensional polyhedron: That is

$$V - E + F = 2$$

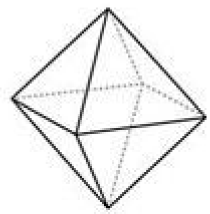
For example, consider the five regular polyhedral: tetrahedron, cube, octahedron, dodecahedron, icosahedron.

The tetrahedron has 4 vertices (V = 4), 6 edges (E = 6) and 4 faces (F = 4): 4 – 6 + 4 = 2

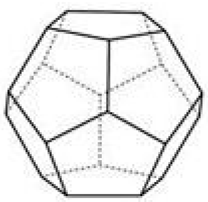




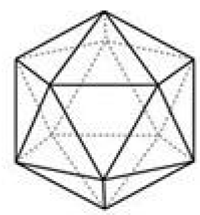
The cube has 8 vertices (V = 8), 12 edges (E = 12) and 6 faces (F = 6): 8 - 12 + 6 = 2



The octahedron has 6 vertices (V = 6), 12 edges (E = 12) and 8 faces (F = 8): 6 – 12 + 8 = 2



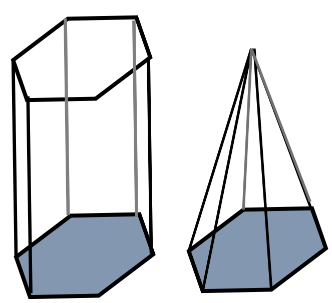
The dodecahedron has 20 vertices (V = 20), 30 edges (E = 3) and 12 faces (F = 12): 20 – 30 + 12 = 2



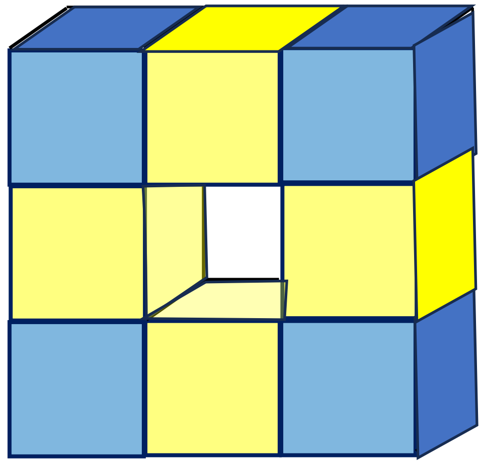
The icosahedron has 12 vertices (V = 12), 30 edges (V = 30) and 20 faces (V = 20): 12 – 30 + 20 = 2

Consider an n-gon *prism* ($n \geq 3$) where two n-sided polygons (e.g. hexagons) make up the ends and the sides are rectangles. $V = 2 \cdot n$, $E = 3 \cdot n$, and $F = n + 2$. Therefore $2 \cdot n - 3 \cdot n + n + 2 = 2$.

A n-gon *cone* with a n-sided polygon base and n triangles coming to a common point has $V = n + 1$, $E = 2 \cdot n$, and $F = n + 1$. Therefore $n + 1 - 2 \cdot n + n + 1 = 2$



Now consider a square donut, a polyhedron with a hole in the middle. One way to visualize it is to think of a 3 x 3 array of cube with the middle cube missing.



Seen from head-on, the front face has 8 faces, 16 vertices, and 24 edges. The back face has the same. Connecting the front and back faces are 16 edges and 16 faces. All together there are 32 vertices, 64 edges and 32 faces.

V – E + F = 32 – 64 + 32 = 0

What this says is a square donut (with a single hole) or any donut with a hole is *topologically different* from a regular polyhedron with no hole – like a muffin.

Or topologist (again the old joke) is a mathematician who can't tell the difference between a donut and a coffee cup.

"Multifaceted Eulerian form
partitioned into infinite
vertices, edges, faces
yielding two still points
V - E + F = 2"

Elegy for a Slide Rule

Fumbling for paper clips in the back of a drawer,
my mottled fingers exhume an old slide rule
upturned from its accidental resting place of forty years.

A scrap of time, chalk frosted blackboards,
professors with crooked glasses and unruly hair,
classmates doodling caricatures of Nixon or Mary Tyler Moore.

I greet my former self, head down, plowing through beautiful years,
clicking the slipstick back and forth with smug precision,
an urgent puritan youth ignorant of tolerances.

Today the dingy cursor still tracks along the bamboo spine,
remnant of talc on the slide, hairline in place but blurred,
unlike crisp new screens flashing answers perfect to the nth degree,

remarkable machines programmed with artificial click of keys,
conjuring immaculate graphs from a morass of pixels and data entries,
their electronic rigor well suited to the previous me.

Cellophane-blue hands stuff the old instrument back into its ossuary,
someday to be reburied in a landfill alongside warped rulers, cracked
protractors, rusted compasses, palpable bones of the mortal body of math.

My self and the slide rule will decompose to respective atoms,
drift in cold space or populate some as yet unborn star,
while the bloodless domain that figures pi beyond the quadrillionths,
exact and flawless, will switch off like a light

-- E R Lutken (3: A Taos Press © 2021)

The Cost of Calculation

Mathematical calculation is costly especially before the advent of positional notation, which greatly simplified calculation as seen in this 1504 work of art by Gregor Reisch where the Spirit of Arithmetic smiles on Boethius who calculates using positional notation (aka Hindu-Arabic numbers) while Pythagoras on the right struggles using a counting board for his calculations.

Note the use of a counting board as an aid to computation – certainly needed if one was restricted to using Roman numerals to represent numbers.

However, while a counting board or abacus was an effective (?) computing device for addition and subtraction since you were only adding or subtracting tokens from various locations, when it came to multiplication its only use was to store partial products that had to be added up. And as for division ….

The fact is that multiplication and division carry a much higher calculational cost than addition and subtraction aside have having to also memorize the multiplication table.

For example, to add 357 and 589

```
 356
+589
----
 945
```

Allegory of Arithmetic - Gregor Reisch - 1504

You add 6 + 9 = 5 carry 1, 1 + 5 + 8 = 4 carry 1, and 1 + 3 + 5 = 9. One pass and you're done.

```
   356
  ×589
  ----
  3204
 2848
1780
------
209684
```

But to multiply 356 by 589 it's 9 times 346 = 3204 plus 8 times 356 = 2848 shifted left one digit because it's really 80 times 356 plus 5 times 356 = 1780 shifted left 2 digits because it's really 500 times 356. Three multiplication passes plus two additions. Whew!

This is why logarithms were (are) important!

John Napier, Henry Briggs, and logarithms

In 1615 Henry Briggs (1561 – 1630), a mathematician from Gresham College in London England journeyed north to Edinburgh Scotland to meet John Napier (1550 – 1617), Laird of Merchiston, who in 1614 published a treatise titled *Mirifici Logarithmorum Canonis Descriptio* describing a new more efficient method for multiplication and division using what is now referred to as *Napierian logarithms*.

As told by John Marr to William Lilly *"Mr Briggs appoints a certain day when to meet at Edinburgh; but failing thereof, Merchiston was fearful he would not come. It happened one day as John Marr and the Lord Napier were speaking of Mr Briggs, "Oh! John," saith Merchiston, "Mr Briggs will not come now"; at the very instant one knocks at the gate, John Marr hastened down and it proved to be Mr Briggs to his great contentment. He brings Mr Briggs into my Lord's chamber, where almost one quarter of an hour was spent, each beholding other with admiration, before one word was spoken. At last Mr Briggs began, - "My Lord, I have undertaken this long journey purposely to see your person, and to know by what engine of wit or ingenuity you came first to think of this most excellent help unto astronomy, viz. the Logarithms …"*-

Common logarithms (a.k.a. Briggsian logarithms) are Henry Briggs's contribution to Napier's work.

Logarithms is a way to multiply (and divide) using addition (or subtraction) – it requires two tables of figures: a table of logarithms which given a number finds its logarithm and a table of anti-logarithms which given a logarithm finds its corresponding number.

To multiply 356×589 you looked up the logarithms of 356 and 589:

$$Log(356) = 2.55144998$$
$$Log(589) = 2.770115295$$

Then you added the two logarithms together:

$$2.55144998 + 2.770115295 = 5.321565293$$

Then using the table of anti-logarithms, you looked up the sum to obtain the product.

$$Log(356) = 2.55144998$$
$$Log(589) = 2.770115295$$
$$2.55144998 + 2.770115295 = 5.321565293$$
$$AntiLog(5.321565293) = 209684$$

Division was done by subtracting instead of adding the two logarithms (since $\frac{10^a}{10^b} = 10^{a-b}$).

So, what is the common logarithm of a number *n*? The logarithm of *n*, *Log(n)*, is the exponent you raise 10 to equal *n*, so $n = 10^{Log(n)}$ Therefore, using the laws of exponents (the mathematics behind logarithms) :

$$n \times m = 10^{Log(n)} \times 10^{Log(m)} = 10^{Log(n)+Log(m)} = anti - Log\left(Log(n) + Log(m)\right)$$

where the *anti-Log(n)* is the is the *inverse* of *Log(n)*.

That is

$$Log(n) = k \ \textit{if and only if} \ anti - Log(k) = n$$

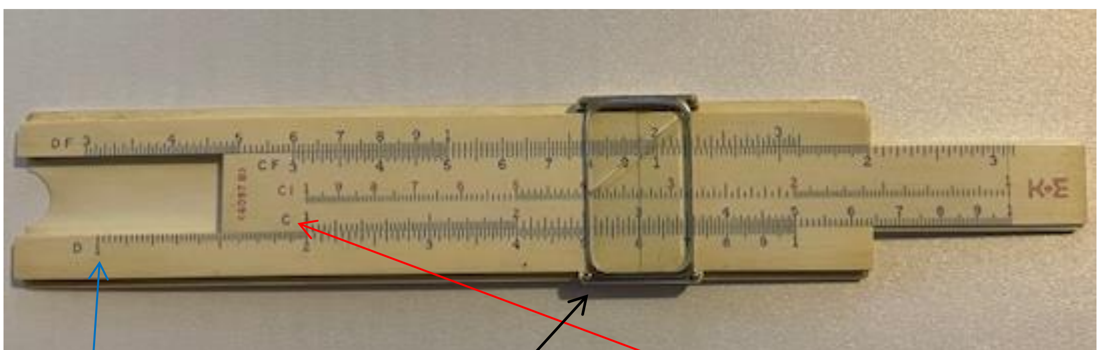
This method reduces multiplication to three table look-ups (two *Log* lookups and one *anti-Log* lookup) and one addition (or one subtraction for division).

Of course, you have to generate (and publish) tables of logarithms and anti-logarithms (that's what Henry Briggs and others did for common logarithms) but this needs only be done once.

Pierre-Simon Laplace observed that logarithms "by shortening the labors doubled the life of astronomers".

Enter the Slide Rule

The slide rule was invented by William Oughtred (1574 – 1660) around 1622 by marking two "rulers" with *logarithmic markings* (instead of equidistant marking normally found on a ruler) and sliding the two together to multiply and divide. That is if we "fix" the distance between the integers 1 and 10 to be a length of one, the distances of the in-between integers 2, 3, 4, 5, 6,7, 8, and 9 are respectively log(2) = 0.30103, log(3) = 0.47712, log(4) = 0.60206, log(5) = 0.69897, log(6) = 0.77815, log(7) = 0.84510, log(8) = 0.90301, and log 9) = 0.95424, their respective logarithmic values (rounded to 5 digits). Note the log(1) = 0.0 since any number *n* raised to the 0 power equals 1 ($n^0 = 1$) and log(10) = 1.0



To multiply 2 times 3 you would position (slide) the "1" on the upper "C" ruler to the "log 2" marking on the lower "D" ruler (distance 0.30103) and then from the "log 3" marking on the upper "C" ruler (distance 0.47712) then using the sliding cursor look down to see the 6 (distance 0.77815 – 0.30103 + 0.47712) – the product on the lower "D" rule. Division was done by reversing the process.

To multiply 356 times 589 you would position the "1" on the upper ruler (marked C on left) over the (approximate) "log 3.56" marking on the lower ruler (marked D on left) then using the sliding cursor scan over to the (approximate) "log 5.89" marking on the upper C ruler and read off the number of the lower D ruler. Since you were multiplying 3.56×10^2 and 5.89×10^2 your answer would be 20.9684×10^4. Using the slide rule required the user to keep track of orders of magnitude; a slide rules might only have an accuracy to 2 or 3 decimal places (though size counts!).

In addition to the C and D scales used for multiplication and division, there were other scales for multiplication/division by π and multiplication/division by trig functions.

"Today the dingy cursor still tracks along the bamboo spine,
remnant of talc on the slide, hairline in place but blurred,
unlike crisp new screens flashing answers perfect to the nth degree,"

In summary, the analog slide rule of yesterday has been replaced by the digital hand calculator.

"remarkable machines programmed with artificial click of keys,
conjuring immaculate graphs from a morass of pixels and data entries,
their electronic rigor well suited to the previous me."

A Footnote: Using my father's Keuffel & Esser Log-Log Duplex Decitrig slide rule (not pictured above) 356 × 589 obtained an answer of 210,000. Since the correct result is 209,684 the absolute error was 316 but the relative error is $\left|\frac{316}{209684}\right| \approx 0.001487$, off by less than 0.2% which is pretty good.

Did Logarithms lead indirectly to the Computer?

There is a story that Charles Babbage, the inventor of the Difference Engines #1 and #2 and later the Analytic Engine, despairing over the numerous errors found in the hand generated mathematical tables used in calculations (like logarithm tables) exclaimed "*I wish to God these calculations had been executed by steam*".

Supposedly the problem of errors in hand generated mathematical tables led to Babbage's invention of the two Difference Engines, calculating machines that would have generated mathematical tables using steam power and then to his programable Analytic Engine, a never-build precursor to today's modern computer. Charles Babbage is often referred to as the "Grandfather of the Computer".

Since any well-behaved function like a logarithmic function could be approximated by a polynomial, Babbage's Difference Engines used the method of divided differences to compute a *polynomial approximation* to a well-behaved function. However, evaluating a polynomial requires computing powers of the variable x which requires multiplication which as mentioned above is an expensive operation. The method of divided differences nicely gets around this problem requiring only the operations of addition and subtraction. Recall from *Math History in a Few Bad Clerihews* that Pascal constructed a calculating machine that could add and subtract; a similar though much improved mechanism was used by Babbage's engines which were also able to print the results for tables of values for a function – like a table of logarithms thus avoiding the errors found in hand generated mathematical tables.

We'll use a simple example to demonstrate how the method of divided differences works and how it was implemented on Babbage's Difference Engine.

x	f(x)	Δf	Δ²f	Δ³f	Δ⁴f
1	4	-0.125	1.25	0.75	0
1.5	3.875	1.125	2.0	0.75	
2	5	3.125	2.75		
2.5	8.125	5.875			
3	14				

We begin with an unknown function y = ***f(x)*** evaluated at the five x-values 1, 1.5, 2, 2.5 and 3 with corresponding y-values 4, 3.875, 5, 8.125 and 14 (in black on the table to the right). The first difference values (in red) in column **Δf** are obtained by subtracting adjacent *f(x)* values, i.e. $f(x_{n+1}) - f(x_n)$. Likewise, the 2nd difference values $\Delta^2 f$ are obtained in a similar manner from the Δf column and so on. Note that the entry for the 4th difference is 0 which suggests that y = *f(x)* might be a cubic polynomial.

The five values on the first row: `4, -0.125, 1.25, 0.75, 0` are the starting values used to recursively compute values for the function *f(x)*. To start each new row of figures first copy the right most column value (0) to the row below. Next moving *right to left* pairwise sum adjacent entries entering each sums in the row below ; that is

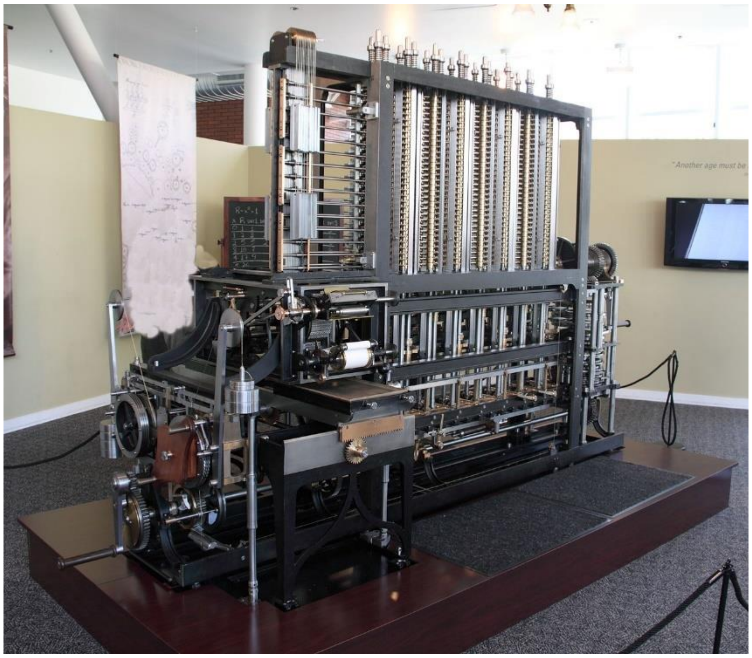
```
| 4.0 + -0.125 | -0.125 + 1.25 | 1.25 + 0.75 | 0.75 + 0 |  0
|   3.875      |     1.125     |     2.0     |   0.75   |  0
```

x	f(x)				
1	4.0	-0.125	1.25	0.75	0
1.5	3.875	1.125	2.0	0.75	0
2	5.0	3.125	2.75	0.75	0
2.5	8.125	5.875	3.5	0.75	0
3	14.0	9.375	4.25	0.75	0
3.5	23.375	13.625	5	0.75	0

Repeating this pairwise summing of adjacent entries, we obtain new and existing entries in rows *2: f(1.5), 3:f(2), 4:f(2.5) and 5:f(3)*. Eventually we compute *new entries* for *f(3.5), f(4), f(4.5)* etc. using a method that *computes a polynomial approximation* to the function *f(x)* – all done using only addition and subtraction.

Babbage's Difference Engine #2 was finally constructed in the 1990's (see picture on the right). The Engine could store eight 31-digit numbers and could generate a table of 7th degree polynomial approximations to any function.

The initial values obtained from the divided difference method were stored in number columns 1 thru 8. The engine added column 8 to column 7, column 7 to column 6, etc. finally adding column 2 to column 1 obtaining the new value of *f(x)* in column 1. Repeating the action of the Difference Engine would result in a list of values for the function *f(x)*. Babbage incorporated a printer that would print out the final values of *f(x)* thus avoiding human mistakes in calculating mathematical tables.



Difference Engine (No 2) - Computer History Museum in Mountain View, CA

An efficient mechanism used by the Difference Engine first added the odd numbered columns to the even numbered columns (7 to 6, 5 to 4, 3 to 2) then the even numbered columns to the odd numbered columns (8 to 7, 6 to 5, 4 to 3, 2 to 1) instead of *rippling the addition* of columns 8 to 7, 7 to 6 etc. down the line of columns. Babbage was a mechanical genius!

The Difference Engines #1 and #2 were never completed (nor was Babbage's Analytic Engine) with Difference Engine #2 being a better version of #1. In the 1990's using Babbage's drawings, two copies of Difference Engine #2 were built thus validating Babbage's original design.

The Happy Ending Problem

for Paul Erdős, George Szekeres, and Esther Klein

For the dots and lines, the cups and caps
required to create convex polygons,
figures with no dents to catch sadness,
no pits to hide festering anaerobes,
no diverticula to sequester pain,
there might be a precise formula
(for an n-sided polygon, $1+ 2^{n-2}$ dots)
but as of now, the answer is elusive.

Diamond lives of shimmering proofs, paintings, poems,
epic loves lasting through lifetimes,
storied octagons, nonagons, decagons, on and on,
astonishing constellations traced in cold stars,
miracles of combinatorics shine in a chaotic world.

In deep space churning with galactic flecks and threads,
shapes melt like snowflakes, each tearful unraveling
a vestige of time spent in a house of endings,
last siege of the bravest of sagas.
The weight of loss brought to bear,
at best, a peaceful, concave sorrow

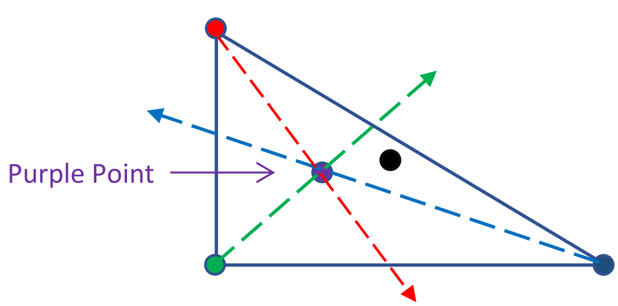
--E R Lutken (3: A Taos Press © 2021)

The Happy Ending Problem

There is a Happy Ending Problem named by Paul Erdos (1913- 1996) which did lead to the marriage of George and Ester (ne Klein) Szekeres which states

> Given any set of 5 points in *general position* in the plane (as opposed to *special position* like a colinear set) there are four points which form a *convex quadrilateral*. It's easy to demonstrate. We may assume the term *general position* means that no three of the points are co-linear.

For example, given 5 points in *general position*, the first three red, green, blue points form a triangle. If a fourth purple point is outside the triangle, a convex quadrilateral is easily obtained so we'll assume the fourth purple point is inside.



A fourth purple point with each of the other three vertex points can be used to construct 3 lines which partition the interior of the triangle into 6 smaller triangles. The fifth black point in any of the 6 small triangles creates an obvious convex quadrilateral. In the above example green-purple-black-blue.

"there might be a precise formula
(for an n-sided polygon, 1 +2^{n-2} dots)
but as of now, the answer is elusive"

This refers to the **Erdős–Szekeres conjecture** which states a more general relationship between the number of points in a general-position point set and its largest convex polygon, namely that the smallest number of points for which any general position arrangement contains a convex set of n points is $1+2^{n-2}$ points. *It's an unproven conjecture*.

For Example, n = 5 for any set of 9 points a convex pentagon will appear; that is, given any set of 9 points in general position, five of them will form a convex pentagon.

To start, arrange 8 points in a general position (i.e. no 3-point lines) such that there are no convex pentagons (think quadrilaterals only). This is the hard part. Now wherever you place a 9^{th} point a convex pentagon will result.

In other words, it's *impossible* to arrange any 9 points in a *general position in the plane* without creating a convex pentagon.

For n = 6 a convex hexagon is created for any set of $1+2^{n-2}\big|_{n=6}=17$ points. Arrange *16 points* in a general position such that no convex hexagons are formed (think pentagons only) – then whenever you place a 17^{th} point a convex hexagon will result.

A Python Programming Appendix

Finding Primes

The following Python program tests if any integer *n* up to 10,000 has a prime divisor (not equal to *n*). If *n* has a prime divisor, then *n* is not prime; otherwise *n* is prime.

```
def main():
     plist = [2,3,5,7,11,13,17,19,23,29,31,37,41,43,47,
              53,59,61,67,71,73,79,83,89,97] # list of primes < 101
    n = eval(input("\nEnter an integer less than 10,000: "))
    IsPrime = True   # assume n is prime and test for otherwise
    for p in plist:  # search for a prime divisor
        if (n % p == 0) and (n != p): # if prime p divides n and n is not p
            IsPrime = False           # then n is not prime
            break
    if IsPrime: # found no prime divisors
        print("\n{0} is Prime".format(n))
    else:       # found a prime divisor
        print("\n{0} is not prime with divisor {1}".format(n,p))
```

MonteCarloTruel.py

Below is Python code (and sample run) for a simulation of a Truel with some simplifications. In each round, a remaining gunfighter can randomly shoot any other gunfighter thus allowing both *duel plus one* and *round robin* configurations to occur randomly.

The variables `g1`, `g2`, and `g3` are 0 or 1 indicating whether Gunfighter #1, Gunfighter #2 and/or Gunfighter #3 are dead (0) or alive (1). Their sum is used to track how many gunfighters are left standing. The variables `p1`, `p2`, and `p3` are the corresponding *probabilities* for each gunfighter being able to hit his or her target *assuming the same probability for each target*. *All surviving gunfighters shoot simultaneously*.

`coinToss()` returning `'H'` (heads) or `'T'` (tails) is used (in `threeWay()` see below) to randomly select which gunfighter is being targeted. That is, each $s_{i,j} = 0.5$

`hit(p)` returns `True` or `False` indicating if that gunfighter (indicated by parameter `p1`, `p2`, or `p3`) was successful in hitting his or her target, If successful `coinToss()` then is used in the `treeWay()` function (see below) to select the targeted gunfighter.

`threeWay(p1,p2,p3)` randomly allows each gunfighter to shoot at the other two returning 0 or 1 for each targeted gunfighter indicating if they are alive (1) or dead (0). Action is simultaneous.

`twoWay(p1,p2),twoway(p2,p3), and twoway(p1,p3)` does the same for any pair gunfighters who survive a three-way shoot out.

The variables `cnt0, cnt1, cnt2,` and `cnt3` track the number of no winners, gunfighter #1 wins, gunfighter #2 wins and gunfighter #3.

```python
# File: MonteCarloTruel.py
# Date: September 25, 2021
#
# Desc: Monte-Carlo Simulation of a Truel
#       a three way duel
#

from random import *  # import random number generator functions

def coinToss():
    # returns 'H' or 'T'
    if randrange(2) == 1:
        return 'H'
    else:
        return 'T'

def hit(p):
    # returns True if random value <= p
    return random() <= p

def threeWay(p1,p2,p3): # one round - each fires one shot

    g1 = g2 = g3 = 1  # assume all gunfighters are alive

    # g1 fires at g2 or g3
    if hit(p1):
        if coinToss() == 'H':
            g2 = 0 # g2 is hit
        else:
            g3 = 0 # g3 is hit

    # g2 fires at g1 or g3
    if hit(p2):
        if coinToss() == 'H':
            g1 = 0 # g1 is hit
        else:
            g3 = 0 # g3 is hit

    # g3 fires at g1 or g2
    if hit(p3):
        if coinToss() == 'H':
            g1 = 0 # g1 is hit
        else:
            g2 = 0 # g2 is hit

    return g1, g2, g3    # return who lives/who dies

def twoWay(p1,p2):  # one round - each fires one shot

    g1 = g2 = 1# assume both gunfighters are alive

    if hit(p1):
        g2 = 0
    if hit(p2):
        g1 = 0
    return g1, g2 # return who lives/who dies
```

```
A Sample Run

Monte-Carlo Based Truel

Input #1 Probability: 0.5
Input #2 Probability: 0.5
Input #3 Probability: 0.5
Input Number of Trials: 50000

Total Outcome Counts

#1 Survives: 13079 - 26.16%
#2 Survives: 13150 - 26.30%
#3 Survives: 13055 - 26.11%
No Survivors: 10716 - 21.43%
```

```python
def main():
    print("\nMonte-Carlo Based Truel\n")

    # get probabilities for a hit

    p1 = eval(input("Input #1 Probability: "))
    p2 = eval(input("Input #2 Probability: "))
    p3 = eval(input("Input #3 Probability: "))

    numTrials = eval(input("Input Number of Trials: "))

    # initiallize count of winners

    cnt0 = 0  # count no survivors
    cnt1 = 0  # count of #1 survives
    cnt2 = 0  # count of #2 survives
    cnt3 = 0  # count of #3 survives

    for k in range(numTrials):  # main counting loop

        g1 = g2 = g3 = 1   # 0 = dead; 1 = alive

        while (g1 + g2 + g3 == 3): # loop while 3 survivors
            g1, g2, g3 = threeWay(p1,p2,p3)

        while (g1 + g2 + g3 == 2):  # loop while two survivors
            if g1 == 0:  # 2 and 3 survived
                g2, g3 = twoWay(p2,p3)
            elif g2 == 0:  # 1 and 3 survived
                g1, g3 = twoWay(p1,p3)
            elif g3 == 0:  # 1 and 2 survived
                g1, g2 = twoWay(p1,p2)

        # total results

        cnt1 = cnt1 + g1
        cnt2 = cnt2 + g2
        cnt3 = cnt3 + g3
        if (g1 + g2 + g3) == 0:
            cnt0 = cnt0 + 1

    # output results

    print("\nTotal Outcome Counts\n")
    print("#1 Survives: {0:4} - {1:5.2f}%".format(cnt1,cnt1/numTrials *100))
    print("#2 Survives: {0:4} - {1:5.2f}%".format(cnt2,cnt2/numTrials *100))
    print("#3 Survives: {0:4} - {1:5.2f}%".format(cnt3,cnt3/numTrials *100))
    print("No Survivors: {0:4} - {1:5.2f}%".format(cnt0,cnt0/numTrials *100))
    print()

main()
```

An Afterword: Primes. the Palindromic Year 2002 and Discovery of Patterns

Mathematics is often described as the discovery of *numeric patterns*. This is a very human activity since we constantly detect *patterns* all around us, *numeric or not* so in some sense all of us, as humans, are mathematicians. For example, consider the pattern found in the year 2002 – it's a palindrome number reading the same forward and backwards. Mathematics seeks to understand the *logical necessity for why these patterns exist*. For example, there are only ten palindromic years for the 3rd millennium. To make this statement we do not need to check out all one thousand integers between 2000 and 2999. Instead, we look deeper. We know that the first and last digits both must be 2 and we know that the middle two digits are the same. Since there are only ten digits there can only be ten palindromic years. This is mathematics: the discovery of a pattern and the presentation of a logical argument demonstrating why the pattern necessarily follows.

The fascination of mathematics is that when one pattern is solved, another immediately presents itself which leads to the discovery of a deeper pattern. Since we know about primes (positive integers whose only divisors are one and themselves) it is natural to ask if an integer is prime. There is a logical argument (we call them *proofs*) that shows that any palindromic number with an even number of digits has 11 as a divisor. Therefore, every even digit palindrome is *composite* (non-prime). However, this proof does not hold for palindromes with an odd number of digits as there are odd digit palindromes which are prime (example 929). We know there is no largest prime but is there a largest *prime palindrome*?

Mathematics is important because of its usefulness in solving *real world problems*. (Recall Eugene Wigner's statement about the *unreasonable effectiveness of mathematics* from the opening *Fundamentals of Mathematics* essay). But that's only half of the fascination of mathematics, and in my opinion, the uninteresting half. Mathematics provides us with tools to understand patterns found in everyday life. Mathematics helps us to define these patterns, understand these patterns, and then leads us to discover deeper patterns. The fascination of mathematics is both universal and timeless. As the fifth century philosopher Proclus put it

This, therefore, is mathematics; ... she gives light to her own discoveries; she awakens the mind and purifies the intellect; she brings light to our intrinsic ideas; she abolishes the oblivion and ignorance which are ours by birth.