Pi to 10,000 Digits

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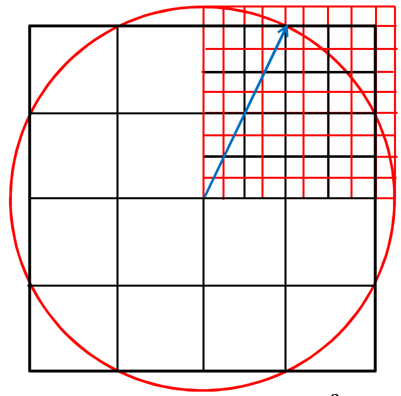
As part of an investigation on how, in 1949, the ENIAC computed the decimal expansion of pi to 2035 digits, I had to first determine how to do the calculations on a modern computer before understanding how it could be done on the ENIAC's very different architecture. It required high precision calculations on numbers more than 2000 digits long. This talk will present how to do the calculations to determine pi to 10000 (or more) digits – in case you might want to try it for yourself.

1. Rhind Papyrus (1650 BCE)

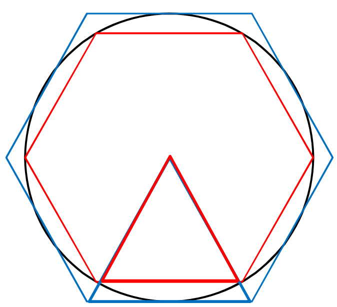
conjectured by H. Engels



$$Area \approx \left(\frac{8}{9}Diameter\right)^2$$

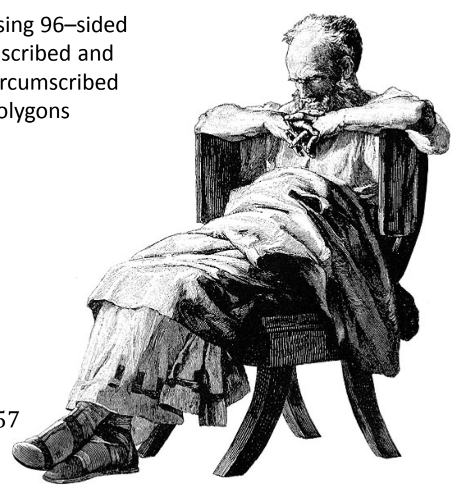
$$\pi \approx \left(\frac{16}{9}\right)^2 \approx 3.1605$$

2 - Archimedean Methods



using 96–sided inscribed and circumscribed polygons

$$3.140845 \approx 3\frac{10}{71} < Area < 3\frac{1}{7} \approx 3.142857$$



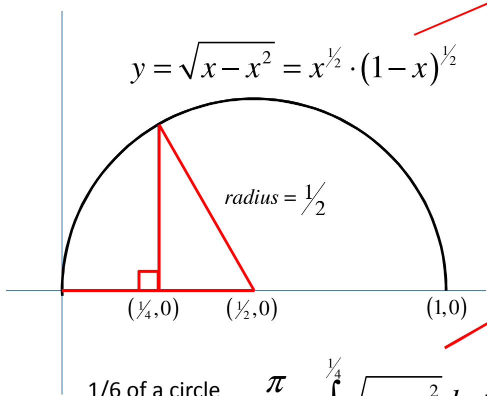
Detail of an engraving by M. Weber of a painting by the Italian artist Niccolò Barabino (1832-1891).

3. Newton's Pi (*Methodus Fluxionum et Seriersum Infinitarum* – written 1671)

$$x^{1/2}\left(1-\frac{1}{2}x-\frac{1}{8}x^2-\frac{1}{16}x^3-\frac{5}{128}x^5-..\right)$$

expand by binomial theorem

$$y=\sqrt{x-x^2}=x^{1/2}\cdot(1-x)^{1/2}$$



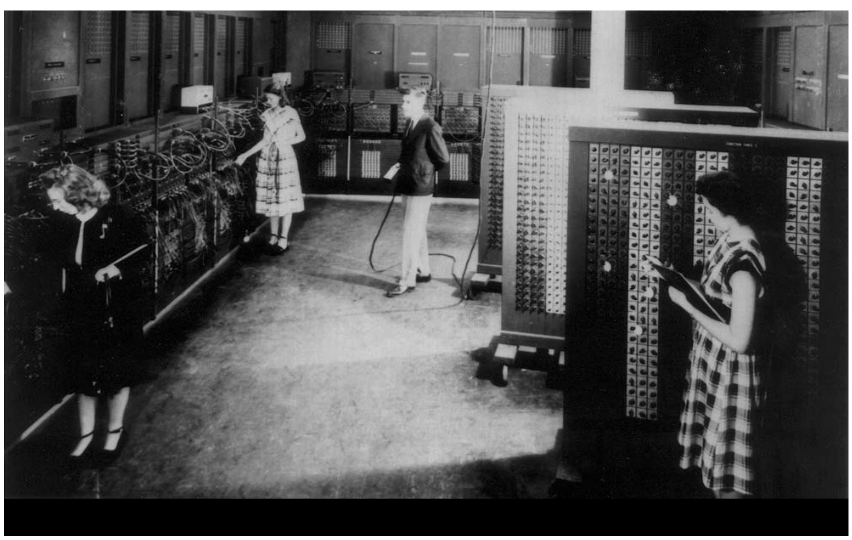
area of the 30-60-90 triangle

1/6 of a circle of radius 1/2

$$\frac{\pi}{24}=\int_0^{1/4}\sqrt{x-x^2}\,dx+\frac{\sqrt{3}}{32}$$

Dunham p. 174

4. Enter "The Computer" (1949)



ENIAC at BRL: (*left to right)* Homé McAllister, Winifred (Wink) Smith, George Reitwiesner, and Ruth Lichterman

Arctan Series for Computing π

Arctan Formula

$$\arctan(x) = \int_0^x \frac{1}{t^2+1}\,dx = \sum_{k=0}^{\infty} (-1)^k \frac{x^{2k+1}}{2k+1}$$

Gregory-Leibniz Series

$$\frac{\pi}{4} = \arctan(1) = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \ldots$$

converges *very* slowly

Machin's Formula

$$\frac{\pi}{4} = 4\arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

John Machin 1680 – 1751

Specifically the two arctangent series

$$\arctan\left(\frac{1}{5}\right) = \frac{1}{5} - \frac{1}{3\cdot 5^3} + \frac{1}{5\cdot 5^5} - \frac{1}{7\cdot 5^7} + \frac{1}{9\cdot 5^9} - \ldots$$

$$\arctan\left(\frac{1}{239}\right) = \frac{1}{239} - \frac{1}{3\cdot 239^3} + \frac{1}{5\cdot 239^5} - \frac{1}{7\cdot 239^7} + \ldots$$

Notice how each term can be computed using the preceding term

The Mechanics of High Precision Calculation
10,000 Digits

Representing Numbers
Division
Addition & Subtraction (Carries and Borrows)
Multiplication by 4

Representation: arrays of integers – 5 digit per component

```
1/5^1   = 00000.20000 00000 00000 …
1/5^3   = 00000.00800 00000 00000 …
1/5^5   = 00000.00032 00000 00000 …
1/5^7   = 00000.00001 28000 00000 …
1/5^9   = 00000.00000 05120 00000 …

1/239^1 = 00000.00418 41004 18410 …
1/239^2 = 00000.00001 75066 96311 …
1/239^3 = 00000.00000 00732 49775 …
```

Extended precision division (left to right)

```
1/239¹ = 00000.00418 41004 18410 …

          1 r 179             75066 r 230
239/00418           239/17941004

      96311 r 81              33908 r 172
239/23018410        239/ 8104184

1/239² = 00000.00001 75066 96311 …
```

Extended Addition & Subtraction

± Right to Left with Propagated Carries & Borrows

```
+1/5¹       = +0.20000 00000 00000 00000 …
-1/(3×5³)   = -0.00266 66666 66666 66666 …
              ---------------------------
              +0.19733 33333 33333 33333 …
+1/(5×5⁵)   = +0.00006 40000 00000 00000 …
              ---------------------------
              +0.19739 73333 33333 33333 …
```

Example with first 3 terms of arctan(1/5) series

Putting the arctan(1/5) calculations all together

```
1/5¹        = +0.20000 00000 00000 00000 …
SUM         = +0.20000 00000 00000 00000 …
1/5²        = +0.04000 00000 00000 00000 …
1/5³        = +0.00800 00000 00000 00000 …
-1/(3×5³)   = -0.00266 66666 66666 66666 …
SUM         = +0.19733 33333 33333 33333 …
1/5⁴        = +0.00160 00000 00000 00000 …
1/5⁵        = +0.00032 00000 00000 00000
+1/(5×5⁵)   = +0.00006 40000 00000 00000 …
SUM           +0.19739 73333 33333 33333 …
```

Details of example with first 3 terms of arctan(1/5) series

Overview – Computing π to 10000 digits: Four Steps

arctan(1/5)

+0.19739 55598 49880 75837 …

arctan(1/239)

−0.00418 40760 02074 72386 …

$\pi/4 = 4 \times \arctan(1/5) - \arctan(1/239)$

+0.78539 81633 97448 30962 …

Left to right multiply by 4 with carry

$\pi = 4 \times \pi/4$

+3.14159 26535 89793 23846 …

Accuracy & Precision

Arctan(1/5) converges slowest so solve

$$\frac{1}{(2k+1)\cdot 5^{2k+1}} < \frac{1}{10^{10000}}$$

$$10^{10000} < (2k+1)\cdot 5^{2k+1}$$

$$\log\left(10^{10000}\right) < \log\left((2k+1)\cdot 5^{2k+1}\right)$$

$$10000 < \log(2k+1) + (2k+1)\log(5)$$

Using Newton's Method k ≈ 7150

Beyond 10,000 Digits of Precision?

The range of a 32-bit integer is ≈ ±2,147,483,647

k ≈ 7150 means that dividends are in the neighborhood of 1,430,000,000 – so we're pushing the size of representable integers.

π: Some (Early) Computer Generated Milestones

1949: ENIAC - 2035 digits (70 Hours)

1953: NORC – 3089 digits (13 minutes)

1958: IBM 704 – 10,000 digits (1 hr. 40 min.)

1959: IBM 704 – 16,167 digits (4.3 hours)

1961: IBM 7090 – 100,000 digits (8 hr. 43 min.)

$$\pi = 24\arctan\left(\frac{1}{8}\right) + 8\arctan\left(\frac{1}{57}\right) + 4\arctan\left(\frac{1}{239}\right)$$

October 16, 2011 – 10,000,000,000 digits (371 days)

http://www.numberworld.org/misc_runs/pi-10t/details.html

Why?

"Early in June, 1949, Professor John von Neumann expressed an interest in the possibility that the ENIAC might be employed to determine the value of π and e to many decimal places with a view towards obtaining a statistical measure of the randomness of distribution of the digits"

"Only the following minor observation is offered at this time concerning the randomness of the distribution of digits … A preliminary investigation has indicated that the digits of e deviate significantly from randomness (in the sense of staying closer to their expected values than a random sequence of this length normally would) while for π no significant deviations have so far been found" – G.W. Reitweisner, *A ENIAC Determination of π and e*

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Thank You

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