Multiplication and Division using Roman Numerals
November 2020

There is a famous graphic titled "Allegory of Arithmetic" by Gregorius Reich (1503) that depicts a contest between Boethius (a 5th century philosopher) using Hindu-Arabic numbers and Pythagoras using a counting board to solve an arithmetic problem. Over and above them is Arithmetica who smiles down at Boethius giving her approval to the use of Hindu-Arabic numbers for calculation.

As any student knows, it is not clear now one can do arithmetic using Roman Numerals directly, especially as everyone learns to do hand-calculation using so-called Hindu-Arabic numbers, which does require memorizing 10 by 10 addition and multiplications tables.

Though while not particularly useful, but it is possible (and fun?) to do multiplication and division directly using Roman Numerals. The technique is based on a method referred to as Russian Peasant Multiplication

Roman Numeral	M	D	C	L	X	V	I
Value	1000	500	100	50	10	5	1

We will not use the so-called subtractive notation; that is 4 is represented by IIII and not IV. And it will be noted below that the bi-quinary (2 and 5 as VV = X and IIIII = V etc.) system of Roman Numeral representation facilitates certain types of calculations.

Before beginning two type of manipulation techniques should be mentioned – Unpack and Pack

Unpack converts a Roman Numeral into 2 or 5 multiples of the next lower valued Roman Numeral.

Example: To unpack X replaces it with VV
To unpack V replaces it with IIIII

Pack consolidates 2 or 5 multiple Roman Numerals into the next higher value Roman Numeral

Example: To pack XXXXX replace it with L
To pack LL replace it with C

Addition and Subtraction

Making use "pack" it's not difficult to add Roman Numerals - you sum respective columns of symbols (I's, V's X's etc.) and then moving right to left you "pack" symbols.

Example:

```
87 =  LXXXVII         L  XXX     V II
139 = CXXXVIIII     C    XXX     V IIII
                    ----------------------
                    C L  XXXXXX  VV  IIIIII   sum
                    C L  XXXXXX  VVV I        pack IIIII to V
                    C L  XXXXXXX V   I        pack VV to X
                    C LL XX      V   I        pack XXXXX to L
                   CC    XX      V   I       pack LL to C       = 226
```

That is *sum* in separate columns the I’s V’s X’s L’x, C’s, D’ and M’s then “pack” (alternating groups of 5’s and 2’s) right to left.

Subtraction may require the use of unpack and since negative Roman Numerals do not exist, a smaller number can only be subtracted from a larger. Again working right to left (least significant to most significant) unpack the minuend until each minuend numeral is greater than or equal to the corresponding subtrahend numeral.

Example

```
81 = LXXXI       L   XXX        I        minuend
47 = XXXXVII         XXXX   V   II       subtrahend
                ----------------
                L   XX    VV   I         unpack X to VV
                    XXXX   V   II
                ----------------
                L   XX    V    IIIIII   unpack V to IIIII
                    XXXX  V    II
                ----------------
                    XXXXXXX V  IIIIIII  unpack L to XXXXX
                    XXXX    V  II
                ----------------
                        XXX    IIII     = 34
```

Doubling and Halving

Because of the “2 & 5 grouping” it is easy to double and halve Roman Numerals.

For example, doubling done by adding the same

```
37 = XXXVII         XXX   V  II
                    XXX   V  II
                    -----------
                 XXXXXX  VV  IIII   sum
                XXXXXXX      IIII   pack VV to X
            L         X      IIII   pack XXXXX to L    = 64
```

Halving is almost as easy. Working from left to right (most significant to least)…

If the number of M’s, C’s and X’s is even – do nothing; otherwise if odd unpack an M, C, or X respectively into a pair of D’s, L’s, or V’s.

If the number of D’s, L’s and V’s is even do nothing; otherwise if odd unpack a D, L, or V into five C’s, X’s or I’s.

Repeat until the number of D’s, C’s, L’s, X’s and V’s is even. The number of I’s may be odd

Now cross out half of each of the remaining numerals. Note if the number of I’s is odd, the throw out the remaining I as the remainder.

Examples

```
48 = XXXXVIII   XXXX  V  III
                XXXX     IIIIIIII  unpack V to IIIII
                XX       IIII      cross out half       = 24

165 = CLV      C      L     X       V
                    LLL     X       V            unpack C to LL
                    LL     XXXXXX   V            unpack L to XXXXX
                    LL     XXXXXX      IIII I    unpack V to IIIII
                     L     XXX         II   r I  cross out half
                                                 = 82 r 1
19 = XVIII    X     V     IIII
                    VVV   IIII        unpack X to VV
                    VV    IIII IIIII  unpack V to IIIII
                     V    IIII rI     cross out half    = 9 r 1
```

Russian Peasant Multiplication (and Division)

This works by forming three columns of numbers – the multiplicand (to be doubled), the multiplier (to be halved) and the product (to be accumulated).

Example: Compute 17×13

```
Multiplicand Multiplier Product
     17         13          0
---------------------------------
     17          6 r 1      0 + 17
     34          3
     68          1 r1       17 + 68 = 85
    136          0 r1       85 + 136 = 221
```

That is – double the multiplicand and halve the multiplier making note of remainders. For each remainder equal to 1 (when an odd multiplier is halved) add the corresponding doubled multiplicand to the product.

So why does this work?

Repeatedly halving 13 can be interpreted as rewriting 13 in terms of powers of 2

That is   – specifically

$$13 = 2^3 + 2^2 + 2^0 = 8 + 4 + 1$$

So $13 \times 17 = \left(2^3 + 2^2 + 2^0\right) \times 17 = 17 \times 2^3 + 17 \times 2^2 + 17 \times 2^0$ which is 17plus 17 doubled twice (68) plus 17 double 3 times (168).

$$13 = 2 \times 6 + 1 = 2 \times (2 \times 3) + 1 = 2 \times \big(2 \times (2 \times 1 + 1)\big) + 1 = 2 \times \Big(2 \times \big(2 \times (2 \times 0 + 1) + 1\big)\Big) + 1$$

```
Multiplicand         Multiplier    Product
    XVII                XIII
-----------------------------------------
                    XIII = VVIII
  XVII =            VVIII ÷ II =
                       VI rI         XVII

                    VI = IIIIII
  XXVVIIII =        IIIIII ÷ II =
  XXXIIII                 III        XVII

  XXXXXXIIIIIIII =    III ÷ II =     LXXVVIIIII =
  LXVIII                I rI         LXXXV

  LLXXVVIIIIII =      I ÷ II =       CLXXXXXXXVVI =
  CXXXVI                  rI         CCXXI
```

Division being the inverse of multiplication works by subtracting out doublings of the divisor while keeping track of the corresponding *value of the power* of 2 (i.e. 2 to the power of the number of doublings)

Division: Compute 72 ÷ 13

Double divisors until you exceed Dividend. The Doubling Exponent (for lack of a better name) is the corresponding *doubling value* (again for lack of a better name) which is always a power of 2

```
 Divisor Doubling
         Exponent
------------------
    13      1
    26      2
    52      4
    104     8
```

Working backward subtract Divisors from Dividend and wherever possible summing the corresponding Doubling Exponent to compute the Quotient. Any "Dividend" left over is the Remainder

```
Divisor Dividend Doubling Quotient
           72    Exponent
----------------------------------
  104               8
   52      20       4         4
   26               2
   13       7       1         5
                           -----
                             5 r 7
```

This works because you are subtracting out powers of 2 times the divisor

$$72 = 52 + 13 + 7 = 13 \times 2^2 + 13 \times 2^0 + 7 = 13 \times \left(2^2 + 2^0\right) + 7 = 13 \times 5 + 9$$

```
  Divisor       Doubling
                Exponent
------------------------
    XIII          I

    XXIIIIII =   II
    XXVI

    XXXXVVII =   IIII
    LII

    LLIIII =     VIII
    CIIII
```

Compute LXXII ÷ XIII

```
Divisor    Dividend        Doubling    Quotient
            LXXII          Exponent
-----------------------------------------------
   LII       LXXII - LII =   IIII        IIII
             XX
   XXVI

   XIII      XX - XIII =       I         IIIII =
             XVV - XIII =                V
             XVIIIII - XIII =
             VII
```

```
Therefore LXXII ÷ XIII = V r VII
```

Finally, no claim is being made that this is superior to our common everyday use of our 10-digit positional notation. The obvious drawback to this is the limit range of Roman Numbers (though the placement of bar over the symbols V thru M (that is $\overline{V}$, $\overline{X}$, $\overline{L}$, $\overline{C}$, $\overline{D}$, $\overline{M}$) allowed the representation of values up to 1 million. It does alleviate the need to memorize 10 by 10 addition and multiplication tables and with little practice packing and unpacking, doubling and halving Roman Numerals is easily done.

It's main advantage (?) is that it's a good (?) trick!