The Schroder-Bernstein Theorem

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Schroder-Bernstein Theorem: Given two (infinite) sets A and B, if $|A| \le |B|$ (i.e. there is a one to one mapping from A into a subset of B) and if $|B| \le |A|$ (i.e. there is a one to one mapping of B into a subset of A), then $|A| = |B|$ (i.e. there is a one to one mapping of A onto B).

Note: If either f or g were onto, trivially we are done. If A and B are finite sets then both f and g must be onto. For infinite sets where neither f nor g are onto, the Schroder –Bernstein Theorem is needed to establish a one to one correspondence between A and B.

Proof: This is what we know:

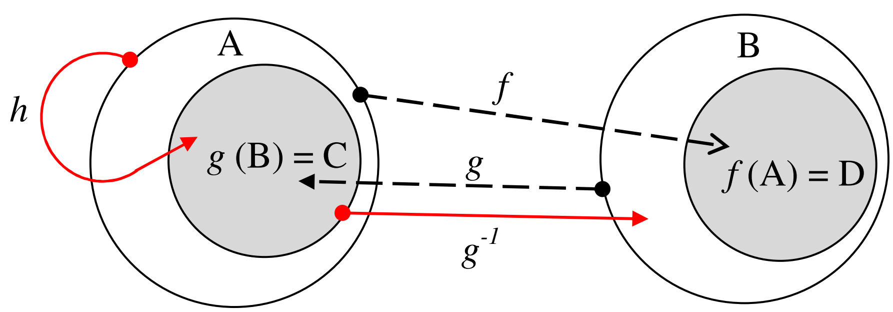
There is a one to one mapping $f : A \to B$ and

there is a one to one mapping $g : B \to A$.

The Approach

Let $C = g(B)$, the range of g. Since $g : B \to C$ is a one to one correspondence (recall that a one to one *correspondence* is both one to one and *onto*), $g^{-1} : C \to B$ exists and is also a one to one correspondence (a function is one to one and onto if and only if it has a inverse which is one to one and onto).

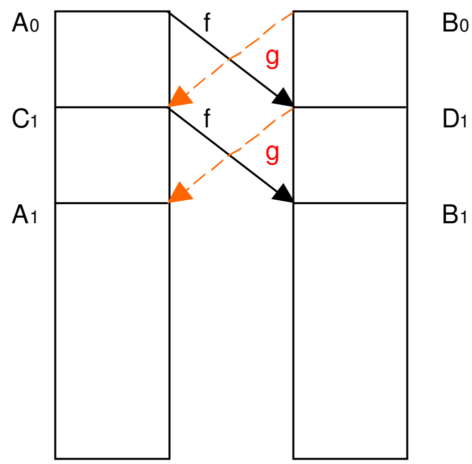
The Schroder-Bernstein theorem works by constructing a *one to one correspondence* between A and its subset $C = g(B)$. If we can construct a one to one correspondence $h : A \to C \subseteq A$, then the composition $g^{-1} \circ h : A \to B$ is the needed one to one correspondence between A and B



1. Use functions f and g to decompose A and B into a sequence of nested subsets

Begin by re-labeling A as A_0, B as B_0

Define $C_1 = g(B_0)$ making $g : B_0 \to C_1$ a one to one correspondence. Set containment $C_1 \subseteq A_0$ follows trivially.



Define $D_1 = f(A_0)$ making $f : A_0 \to D_1$ a one to one correspondence. The containment $D_1 \subseteq B_0$ follows trivially.

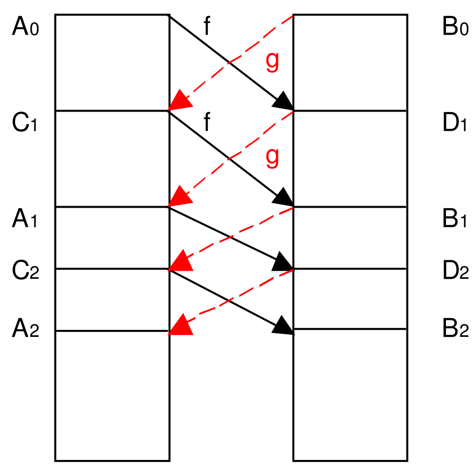
Define $A_1 = g(D_1)$ making $g : D_1 \to A_1$ a one to one correspondence. Since $D_1 \subseteq B_0$ then $A_1 = g(D_1) \subseteq g(B_0) = C_1$ or $A_1 \subseteq C_1$

Define $B_1 = f(C_1)$ making $f : C_1 \to B_1$ a one to one correspondence. Since $C_1 \subseteq A_0$ then $B_1 = f(C_1) \subseteq f(A_0) = D_1$ or $B_1 \subseteq D_1$.

The diagram should make this clear. Set A_1 includes everything in the rectangle from the label down, set C_1 includes everything in the rectangle from the label down (including set A_1) and A_0 is the entire rectangle. The same is true for sets B_1, D_1 and B_0.

We continue the construction of nested subsets to the next "level".

Define $C_2 = g(B_1)$ making $g : B_1 \to C_2$ g a one to one correspondence. Since $B_1 \subseteq D_1$ then $C_2 = g(B_1) \subseteq g(D_1) = A_1$ or $C_2 \subseteq A_1$.



Define $D_2 = f(A_1)$ making $f : A_1 \to D_2$ a one to one correspondence. Since $A_1 \subseteq C_1$ then $D_2 = f(A_1) \subseteq f(C_1) = B_1$ or $D_2 \subseteq B_1$.

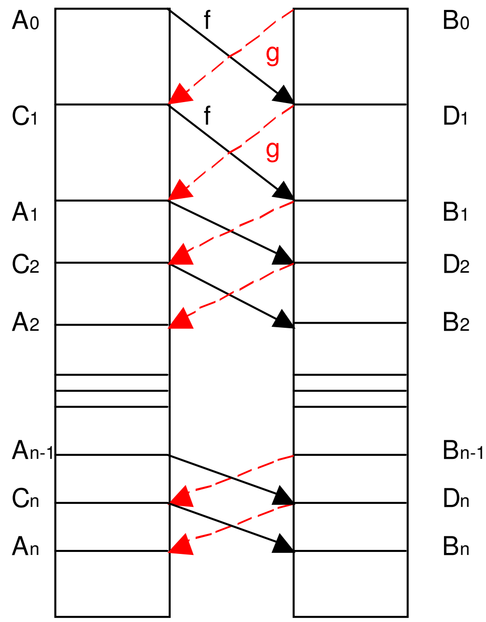
Define $A_2 = g(D_2)$ making $g : D_2 \to A_2$ a one to one correspondence. Since $D_2 \subseteq B_1$ then $A_2 = g(D_2) \subseteq g(B_1) = C_2$ or $A_2 \subseteq C_2$

Define $B_2 = f(C_2)$ making $f : C_2 \to B_2$ a one to one correspondence. Since $C_2 \subseteq A_1$ then $B_2 = f(C_2) \subseteq f(A_1) = D_2$ or $B_2 \subseteq D_2$.

Again the diagram helps to see this.

Using **induction** we continue this process so at the nth step we have

Define $C_n = g\left(B_{n-1}\right)$ making $g : B_{n-1} \to C_n$ g a one to one correspondence. Since $B_{n-1} \subseteq D_{n-1}$ then $C_n = g\left(B_{n-1}\right) \subseteq g\left(D_{n-1}\right) = A_{n-1}$ or $C_n \subseteq A_{n-1}$.



Define $D_n = f\left(A_{n-1}\right)$ making $f : A_{n-1} \to D_n$ a one to one correspondence. Since $A_{n-1} \subseteq C_{n-1}$ then $D_n = f\left(A_{n-1}\right) \subseteq f\left(C_{n-1}\right) = B_{n-1}$ or $D_n \subseteq B_{n-1}$.

Define $A_n = g\left(D_n\right)$ making $g : D_n \to A_n$ a one to one correspondence. Since $D_n \subseteq B_{n-1}$ then $A_n = g\left(D_n\right) \subseteq g\left(B_{n-1}\right) = C_n$ or $A_n \subseteq C_n$

Define $B_n = f\left(C_n\right)$ making $f : C_n \to B_n$ a one to one correspondence. Since $C_n \subseteq A_{n-1}$ then $B_n = f\left(C_n\right) \subseteq f\left(A_{n-1}\right) = D_n$ or $B_n \subseteq D_n$.

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Again a diagram helps show this.

2. Construction a one-to-one onto function *h: A -> C₁*

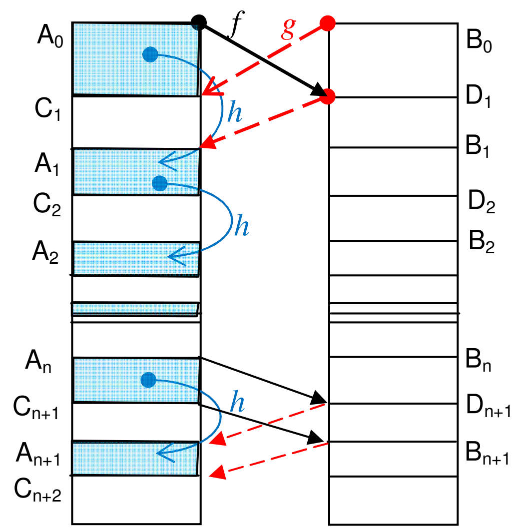
Define h as follows

$$h(x)=\begin{cases} g(f(x)) \text{ if } x\in A_n - C_{n+1} \text{ for some } n \\ x \text{ if } x\in C_n - A_n \text{ or } x\in \bigcap_{n=0}^{\infty} A_n \end{cases}$$

Note that both components of the definition of h are one to one. Moreover from the diagram it's easy to see that the range of h is C_1.

For the first component of $h(x)$ observe that if $x\in A_n - C_{n+1}$ then $h(x)=g\left(f\left(x\right)\right)\in A_{n+1}-C_{n+2}$. This is because for $x\in A_n$ since $f\left(A_n\right)=D_{n+1}$ and $g\left(D_{n+1}\right)=A_{n+1}$ it follows that $h(x)=g\left(f\left(x\right)\right)\in A_{n+1}$.

On the other hand $h(x)\notin C_{n+2}$ because if we assume the opposite, that is $h\left(x\right)\in C_{n+2}$, then since $g^{-1}\left(h\left(x\right)\right)=g^{-1}\circ g\circ f\left(x\right)=f\left(x\right)\in B_{n+1}$ (since g is one to one) then $f^{-1}\left(f\left(x\right)\right)=x\in C_{n+1}$ (since f is one to one) which is a contradiction since $x\in A_n - C_{n+1}$!



So h maps $A_0 - C_1$ onto $A_1 - C_2$ and $A_1 - C_2$ onto $A_2 - C_3$ etc, sort of like Hilbert's Hotel.

The intermediate pieces between, that is $C_1 - A_1$, $C_2 - A_2$, ... , $C_n - A_n$,... etc are mapped to themselves. That is, the second component for h maps $C_1 - A_1$ onto itself, $C_2 - A_2$ onto itself etc.

Finally whatever is left over, the total intersection $\bigcap_{n=0}^{\infty} A_n$ is mapped into itself. Note that if $x\in \bigcap_{n=0}^{\infty} A_n$ then since $C_{n+1}\subseteq A_n$ it follows that $x\in C_{n+1}$ for all n. Therefore $x\notin A_n - C_{n+1}$ for all n and likewise $x\notin C_n - A_n$ for all n. Thus the total intersection $\bigcap_{n=0}^{\infty} A_n$ is disjoint from all sets $A_n - C_{n+1}$ and $C_n - A_n$ (it may be empty!).

3. Establishing a one to one correspondence between A and B

Thus h is a one to one correspondence between A and $C_1 \subseteq A$. Since g^{-1} is a one to one correspondence between C_1 and B, the composition $g^{-1}\circ h : A \to B$ is the sought for one to one correspondence between A and B Thus $|A|=|B|$ and the theorem follows.

QED

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